IMPEDANCE IMAGING, INVERSE PROBLEMS, AND HARRY POTTER'S CLOAK

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Abstract. In this article we provide an accessible account of the essential idea behind cloaking, aimed at nonspecialists and undergraduates who have had some vector calculus, Fourier series, and linear algebra. The goal of cloaking is to render an object invisible to detection from electromagnetic energy, by surrounding the object with a specially engineered "metamaterial" that redirects the energy around the object. We show how to cloak an object against detection from impedance tomography, an imaging technique of much recent interest, though the mathematical ideas apply to much more general forms of imaging. We also include some exercises and ideas for undergraduate research projects.

Key words. electrical impedance tomography, cloaking, metamaterial, Dirichlet-to-Neumann map, Laplace's equation, inverse problem

AMS subject classifications. 35R30, 35-01.

1. Introduction. In the climactic scene to *Harry Potter and the Half-Blood Prince* by J. K. Rowling, a magically petrified Harry watches helplessly as Severus Snape uses the *Avada Kedavra* curse to kill Albus Dumbledore. Harry himself escapes the notice of Draco Malfoy and the other Death Eaters due to the protection of his invisibility cloak, which proves itself invaluable throughout Harry's adventures. The essential property of the cloak isn't simply that it conceals the person underneath—a bed sheet would suffice for that purpose. But by rendering the wearer invisible, the cloak actually conceals the fact that anything at all is being concealed!

Cloaking and invisibility are old staples of popular fiction, especially science fiction, from Romulan ships in Star Trek to the Predator's light-bending armor. The pseudo-explanation usually given is that "the selective bending of light rays" (to quote Mr. Spock) around the object to be cloaked can render the object invisible. But with the laws of physics in the real world, is this possible, even in theory? Physicists and mathematicians have recently found that the answer to this question is a qualified "yes."

The key to cloaking in real life is to engineer a "metamaterial" with special microstructure that bends electromagnetic energy in a quantifiable and controllable way. Scientists and engineers have already made some progress toward designing and constructing metamaterials that successfully cloak objects in certain restricted circumstances. Indeed, cloaking is listed as #2 on New Scientist's top ten list of "sci-fi devices you could soon hold in your hands" [19] and has been a hot topic in other popular science news [9, 22].

In 2006, John Pendry, David Schurig, and David Smith published an idea for a cloak that would render an object in two dimensions invisible to probing by electromagnetic waves at a fixed frequency, by surrounding it with a specially designed metamaterial [21]. Soon afterward, David Smith's group at Duke University constructed a working device based on that idea [23], and in January 2009 reported constructing a device that works for a broad range of frequencies in two dimensions [15]. The same

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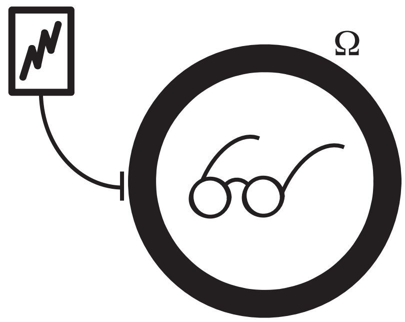

FIG. 2.1. *A region Ω hiding an object from an external observer.*

techniques could, in principle, be scaled to work at optical wavelengths. Greenleaf, Lassas, and Uhlmann [5] had already described essentially the same notion back in 2003, in a study of the inverse problem for electrical impedance tomography posed by Calderón. This group has more recently developed a "double-coating" that can cloak actively radiating sources (e.g., a light source) [6]. For a brief overview of metamaterials and cloaking, see [11]; for in-depth reviews see [7, 8]; and for other approaches to developing cloaks, see [1, 10, 12, 13, 14, 15, 16, 17, 18, 25, 26].

The purpose of this article is to provide an elementary but quantitative, mathematically honest account of the essential idea behind cloaking (following the change of variables method described in [10]), in a way that is entirely accessible to nonspecialists and undergraduates.

2. The Basic Model.

2.1. Electrical Conduction. The goal of cloaking is to render an object invisible, so that even observers who look directly at the object cannot see it. The words "look" and "see" here refer to observers using electromagnetic energy in some form to image objects. The observer might be actively illuminating the object, for example, using radar, or merely making use of ambient energy, like sunlight. It doesn't matter. In this section we'll develop a mathematical model for an electromagnetic imaging technique known as *electrical impedance tomography*, that makes it fairly easy to illustrate the idea behind cloaking. In the next section we'll show how to actually cloak an object so that it is rendered almost or completely invisible to this type of imaging. The techniques apply to much more general imaging methodologies, however.

We might think of the imaging process as taking place in "free space," that is, in $\mathbb{R}^2$ or $\mathbb{R}^3$, but for this exposition it will be simpler to work on a bounded domain Ω. We assume, for convenience only, that Ω is the open unit disk in $\mathbb{R}^2$, and use coordinates (x_1, x_2). We use $\partial\Omega$ to denote the boundary of Ω (the unit circle). Suppose an object is contained in Ω and an external observer attempts to image this object using electromagnetic energy in some form. However, the observer is confined to inject energy, and take measurements, only on $\partial\Omega$. The observer injects electromagnetic energy into Ω, looks at what comes out, and then tries to deduce the interior structure.

In general, one uses Maxwell's equations to quantify the behavior of electromagnetic fields, but this is unnecessarily complicated for our problem. Some simplification could be obtained by modeling the situation with the *wave equation.* A function $u(x_1, x_2, t)$ satisfies the wave equation if $\frac{\partial^2 u}{\partial t^2} - c^2 \triangle u = 0$, where c is the speed of light and $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$, the *Laplacian* operator. For example, the components of the electric and magnetic fields in empty space obey the wave equation. We'll take this simplification one step further, by considering only steady-state or "DC" imaging, in which all quantities are independent of time. Moreover, the interior of Ω (even when "empty") will not consist of empty space, but rather an electrically conductive material. The energy that an observer uses to image the inside of Ω will be electrical current.

Let's start by quantifying what we mean when we say that the interior of Ω is "empty." A material is said to be *homogeneous* if its physical properties are the same at all points, and is said to be *isotropic* if the material has no directional properties. A block of wood, for example, might be (approximately) homogeneous but not isotropic, since the orientation of the grain introduces direction-dependent physical behavior. A material that is not isotropic is *anisotropic*. We will say that the region Ω is "empty" if the interior of Ω is filled with an electrically conductive material that is homogeneous and isotropic with regard to electrical conduction; we assume this is the condition in which an external observer expects to find Ω. Of course, if we place an object inside Ω, the object may not have the same electrical properties, and will alter the way electrical current flows in Ω. This alteration can be used to detect and image the object.

2.1.1. Isotropic Conduction. To quantify all of this, let $u(x_1, x_2)$ denote the electric potential (the "voltage") at the point $(x_1, x_2) \in \Omega$. The electric field $\mathbf{E}$ (a vector field) in Ω satisfies $\mathbf{E} = -\nabla u$. The electric field pushes on conduction electrons and impels a current to flow (though we'll use the "conventional current" model, in which positive charge flows—it doesn't matter). Let $\mathbf{J}$ denote the vector field in Ω that describes the flow of current. The simplest model for how $\mathbf{J}$ depends on $\mathbf{E}$ (and hence u) is

$$\mathbf{J} = \gamma \mathbf{E} \tag{2.1}$$

where γ is the *conductivity*. In the case of a homogeneous isotropic material, γ is simply a non-negative constant, but more generally γ can be a function of position (x_1, x_2) (or in the anisotropic case, a matrix—see Section 2.1.2 below). Equation (2.1) is in some sense just a two-dimensional version of Ohm's Law, and posits a linear relationship between the electric field and current flux, with current always flowing in the direction of $\mathbf{E}$. If γ is large then a lot of current flows for a given electric field strength, whereas when γ is close to zero very little current flows. The extreme case, $\gamma = 0$, corresponds to a perfect insulator—no matter how strong the electric field, no current will flow.

From $\mathbf{E} = -\nabla u$ and equation (2.1) we obtain

$$\mathbf{J} = -\gamma \nabla u. \tag{2.2}$$

If electric charge is conserved in Ω (as it must be if there are no current sources inside) we must have $\nabla \cdot \mathbf{J} = 0$ throughout the interior of Ω. With equation (2.2) this implies

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega. \tag{2.3}$$

In the special case that γ is a constant (that is, when Ω is empty) we can simplify equation (2.3) to *Laplace's equation*

$$\triangle u = 0 \quad \text{in } \Omega, \tag{2.4}$$

where $u = u(x_1, x_2)$. This is the partial differential equation that must be satisfied by the electric potential u inside a homogeneous isotropic conductor. Functions that satisfy equation (2.4) are said to be *harmonic*.

Why is there a (non-constant) potential u inside Ω in the first place? Because an observer has deliberately induced a non-constant potential f on $\partial\Omega$ (e.g., by attaching electrodes to $\partial\Omega$), so that

$$u = f \quad \text{on } \partial\Omega \tag{2.5}$$

for some chosen function f. Equation (2.5) is a *Dirichlet* boundary condition, and f is the *Dirichlet data*.

Laplace's equation (2.4) and the Dirichlet boundary condition (2.5) together constitute a very standard boundary value problem, with a unique solution u for any reasonable (e.g., continuous) applied potential f. But we have not yet accounted for the presence of an object inside Ω, so equation (2.4) is appropriate only for an empty container Ω. In a later section we show how to model and detect the presence of a nonconductive "hole" inside Ω.

Exercise 2.1. *Suppose we parameterize the boundary of the disk in the usual way, as $x_1 = \cos\theta, x_2 = \sin\theta$, $0 \leq \theta < 2\pi$. Let the Dirichlet data at the corresponding point on $\partial\Omega$ be given by $f(\theta) = a\cos\theta + b\sin\theta + c$ for constants a, b, c. Show that the solution to (2.4)-(2.5) is the harmonic function $u(x_1, x_2) = ax_1 + bx_2 + c$.*

2.1.2. Anisotropic Conduction. Many materials exhibit anisotropic physical properties. In the context of electrical conduction, this means that at any given point a material may conduct better in some directions than in others, and so the conduction model of equation (2.1) with γ as a scalar is inappropriate. A natural generalization of (2.1) is to assume that at any given point the material has a direction of maximum conductivity and a direction of minimum conductivity. Let us suppose that the material has maximum conductivity $\gamma_M > 0$ in the direction of the unit vector $\mathbf{v}_M$, and minimum conductivity $\gamma_m > 0$ in the direction of the unit vector $\mathbf{v}_m$, so $0 < \gamma_m \leq \gamma_M$. It's also natural to assume that the direction vectors $\mathbf{v}_M$ and $\mathbf{v}_m$ are orthogonal to each other. A model that captures this behavior is

$$\mathbf{J} = \sigma\mathbf{E} \tag{2.6}$$

where σ is a symmetric positive-definite 2×2 matrix (σ may depend on position), since symmetric positive definite matrices have orthogonal eigenvectors and positive eigenvalues. (Recall that a matrix $\mathbf{A}$ is *positive definite* if $\mathbf{v}^T\mathbf{A}\mathbf{v} > 0$ for all nonzero vectors $\mathbf{v}$, where $\mathbf{v}^T$ is the transpose of $\mathbf{v}$.) The converse is also true: a matrix with an orthogonal basis of eigenvectors and positive eigenvalues is a positive-definite symmetric matrix.

For anisotropic conduction, equation (2.6) replaces (2.1), and equation (2.3) becomes

$$\nabla \cdot \sigma\nabla u = 0 \quad \text{in } \Omega. \tag{2.7}$$

If an electric field $\mathbf{E}$ is applied in a direction that is parallel to $\mathbf{v}_M$ then the resulting current flux is $\mathbf{J} = \sigma\mathbf{E} = \gamma_M\mathbf{E}$, so that $\|\mathbf{J}\| = \gamma_M\|\mathbf{E}\|$. For a fixed magnitude of $\|\mathbf{E}\|$, this direction for $\mathbf{E}$ (parallel to $\mathbf{v}_M$) maximizes $\|\mathbf{J}\|$; see Exercise 2.4 below. Similarly, taking $\mathbf{E}$ parallel to $\mathbf{v}_m$ minimizes $\|\mathbf{J}\|$.

Exercise 2.2. *What 2×2 matrix σ models an isotropic conductor with (scalar) conductivity γ in all directions?*

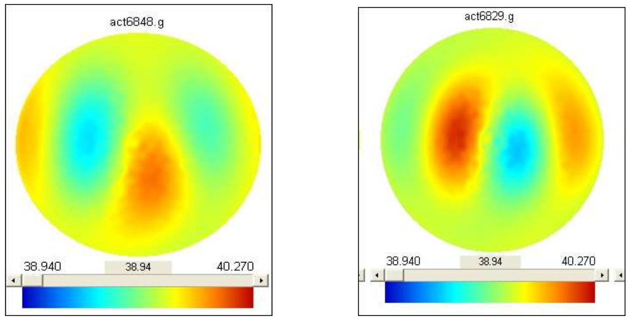

Fig. 2.2. *The image on the left is an impedance image of a cross-section of the torso, taken as blood was filling the subject's heart and leaving the lungs. The area near the heart shows up as red, for the conductivity at this moment is high (blood is very conductive). In contrast, the lungs have little blood in them at this moment and are shown in blue. In the image on the right, the blood has left the heart and entered the lungs, reversing the colors. Our thanks to David Isaacson and the Electrical Impedance Imaging group at the Rensselaer Polytechnic Institute for supplying us with these images, obtained from their ACT III impedance imaging system.*

Exercise 2.3. *Write out an anisotropic conductivity matrix σ to model a homogeneous material with general conductivity γ_M in the direction of the unit vector $\mathbf{v}_M = \frac{\sqrt{2}}{2}\hat{\imath} + \frac{\sqrt{2}}{2}\hat{\jmath}$ and conductivity γ_m in the direction of the unit vector $\mathbf{v}_m = \frac{\sqrt{2}}{2}\hat{\imath} - \frac{\sqrt{2}}{2}\hat{\jmath}$. Hint: use the fact that σ can be diagonalized as $\sigma = \mathbf{PDP}^{-1} = \mathbf{PDP}^T$, where $\mathbf{P}$ is the matrix with the eigenvectors of σ as columns and $\mathbf{D}$ the diagonal matrix of eigenvalues (in the same order as the columns of $\mathbf{P}$.) For a given $\mathbf{E}$, how does the quantity $\sigma\mathbf{E}$ behave as $\gamma_m \to 0^+$?*

Exercise 2.4. *Show that if σ is a symmetric positive definite $n \times n$ matrix and we fix $\|\mathbf{v}\| = 1$ then $\|\sigma\mathbf{v}\|^2$ is maximized when $\mathbf{v}$ is an eigenvector for σ corresponding to the largest eigenvalue(s) for σ. Hint: we can write*

$$\mathbf{v} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$$

for some scalars α_k, where the $\mathbf{v}_k$ are orthonormal eigenvectors for σ; assume that the corresponding eigenvalues are ordered $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Then $\|\mathbf{v}\|^2$ and $\|\sigma\mathbf{v}\|^2$ can be written out quite explicitly in terms of the α_k and λ_k.

2.2. Impedance Imaging. In *impedance tomography* one attempts to image the interior of Ω by applying an electrical current to $\partial\Omega$, say by attaching many electrodes to $\partial\Omega$. The applied current on the boundary induces a spatially-varying potential (i.e., voltage) throughout the interior of Ω, which induces current to flow through the interior. The current must enter or leave Ω through the attached electrodes, and the resulting potential on $\partial\Omega$ (which can be measured) depends on the interior properties of Ω. From this type of information—applied current and resulting voltage—one can deduce information about the interior electrical properties of Ω, such as the conductivity, and thereby form images. See Figure 2.2 for an example of an image of the heart and lungs obtained from an actual impedance imaging system. The article [4] provides a good general overview of the subject.

2.2.1. Imaging Voids. Let's look at how one might image certain special types of objects in Ω with this approach. We take the more mathematically convenient (but equivalent) approach of applying a potential and then measuring the resulting current on $\partial\Omega$. The key is to determine the mapping between the applied potential and the resulting current on the boundary of an object, and how that mapping depends on the interior properties of Ω.

To begin, assume Ω is empty, that is, has homogeneous isotropic conductivity $\gamma > 0$. Suppose we place a non-conductive object "D" inside Ω; think of D a void, that is, missing material. When a potential f is applied to $\partial\Omega$, the presence of the void disrupts the flow of current inside Ω, and this effect should be observable from the boundary. The quantity we will observe is the rate at which electric current flows into Ω at each point on $\partial\Omega$. The rate at which current flows out near a point $p \in \partial\Omega$ is $\mathbf{J}(p) \cdot \mathbf{n}(p)$, where $\mathbf{n}(p)$ is an outward pointing unit normal vector to $\partial\Omega$ at the point p. We will henceforth suppress the dependence of quantities like ∇u or $\mathbf{n}$ on p. Application of equation (2.2) shows that the current flowing out across $\partial\Omega$ at any given point is $-\gamma\nabla u \cdot \mathbf{n}$. The rate at which current *enters* $\partial\Omega$ is thus $\gamma\nabla u \cdot \mathbf{n}$, and is called the *Neumann data* for the function u.

The presence of D in Ω alters the flow of current, for no current can flow into D from $\Omega \setminus D$. This means that $\mathbf{J} \cdot \mathbf{n} = 0$ on ∂D, where here $\mathbf{n}$ denotes a unit normal vector on ∂D, say pointing into D (out of $\Omega \setminus D$). From equation (2.2) we obtain $\gamma\nabla u \cdot \mathbf{n} = 0$ on ∂D. In this case the potential u is defined only in $\Omega \setminus D$ and obeys Laplace's equation there, along with the Dirichlet boundary condition (2.5) on $\partial\Omega$ and the additional boundary condition

$$\gamma\nabla u \cdot \mathbf{n} = 0 \quad \text{on } \partial D. \tag{2.8}$$

Since $\gamma > 0$ we can also write (2.8) as simply $\frac{\partial u}{\partial \mathbf{n}} = 0$, using the shorthand notation $\frac{\partial u}{\partial \mathbf{n}} := \nabla u \cdot \mathbf{n}$ for the normal derivative.

EXAMPLE 2.5. Suppose the observer applies the potential $f(\theta) = \cos\theta + \sin\theta$ to the boundary of the disk. From Exercise 2.1, the resulting potential inside the empty disk is $u(x_1, x_2) = x_1 + x_2$. But if we remove a ball $D = B_{1/2}(0)$ (where we use $B_r(p)$ to denote a ball of radius r centered at the point p) then $u(x_1, x_2) = x_1 + x_2$ is no longer the potential induced in the annulus $\Omega \setminus D$ by the potential f, for u does not satisfy (2.8). To see this, note that ∂D can be parameterized as $x_1 = \frac{1}{2}\cos(\theta), x_2 = \frac{1}{2}\sin(\theta)$, with $\mathbf{n} = -\cos(\theta)\hat{\imath} - \sin(\theta)\hat{\jmath}$. Then

$$\nabla u \cdot \mathbf{n} = (\hat{\imath} + \hat{\jmath}) \cdot (-\cos(\theta)\hat{\imath} - \sin(\theta)\hat{\jmath}) = -(\cos(\theta) + \sin(\theta))$$

which is not identically zero on ∂D. Indeed, in this case the correct potential inside $\Omega \setminus D$ is $u(x_1, x_2) = (x_1+x_2)(4x_1^2+4x_2^2+1)/(5x_1^2+5x_2^2)$. See Figure 2.3 for graphs of the current flux $\mathbf{J}$ for each case. The non-conductive void impedes the flow of the current, and the observer measures (in (r, θ) polar coordinates) $\frac{\partial u}{\partial \mathbf{n}}(1, \theta) = \frac{3}{5}(\cos\theta + \sin\theta)$ on $\partial\Omega$, compared to $\frac{\partial u}{\partial \mathbf{n}}(1, \theta) = \cos\theta + \sin\theta$ for the intact unit disk.

EXERCISE 2.6. *Verify that the function* $u(x_1, x_2) = (x_1+x_2)(4x_1^2+4x_2^2+1)/(5x_1^2+5x_2^2)$ *of Example 2.5 is in fact harmonic on* $\Omega \setminus D$ *(*$D = B_{1/2}(0)$*), with* $u = \cos\theta + \sin\theta$ *on* $\partial\Omega$ *and* $\nabla u \cdot \mathbf{n} = 0$ *on* ∂D*.*

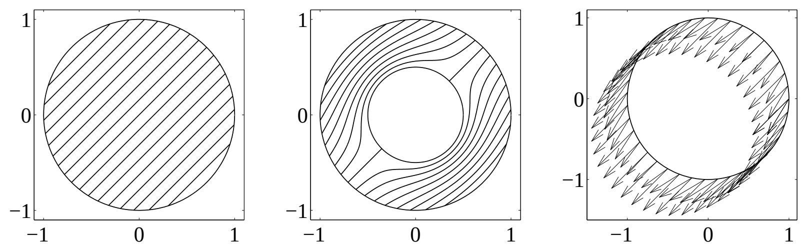

FIG. 2.3. *Comparison of solutions on the unit disk and an annulus. The graph on the left shows flow lines of the current* $\mathbf{J} = -\gamma\nabla u$*, where* u *is the solution to Laplace's equation with Dirichlet condition* $f(\theta) = \cos(\theta) + \sin(\theta)$ *(the potential applied by the observer) on the outer boundary of the unit disk. The flow shown in the middle graph has the same applied potential* f *on the outer boundary plus the Neumann condition of zero flux on the inner boundary of the annulus* $1/2 \leq r \leq 1$*. The rightmost graph compares* $\mathbf{J}$ *on the boundary of the disk and the annulus, where the shorter arrows correspond to* $\mathbf{J}$ *for the annulus.*

2.2.2. Inverse Problems and Cloaking. The above discussion suggests an impedance imaging procedure for gathering information about the interior of Ω (in this case, finding a hole in Ω):

1. Apply a potential f to $\partial\Omega$ (equation (2.5));
2. Measure the response $\gamma\nabla u \cdot \mathbf{n}$ on $\partial\Omega$ (measure the resulting current).

From this kind of "stimulus-response" or Dirichlet-Neumann data we wish to determine the precise size, shape, and location of the hole D. Of course steps (1) and (2) can be repeated with different input potentials f, which might yield additional information. Impedance imaging is an example of an *inverse problem.* The definition of an inverse problem is not set in stone, but might be defined roughly as a problem that requires "deducing cause from effect." In this case, we wish to deduce what interior region D could have yielded the measured boundary current for the applied potential f. Inverse problems of this form often occur in applications where one wants to deduce interior structure from exterior measurements.

We also have at hand the beginnings of a crude cloak. If we want to hide a conductive object inside Ω, we merely excavate a non-conductive hole of some radius $\rho > 0$ in Ω and place the object inside. The object is thus electrically insulated from the outside world and cannot be seen with impedance imaging. Unfortunately the hole itself can be seen, so an observer will know that something is being hidden, even if he can't tell what it is. This would be like Harry Potter substituting a bed sheet for his cloak!

However, the idea of excavating a hole into which we can place something is the beginning of a viable cloak, but first we need to analyze Laplace's equation on an annulus $\Omega/B_\rho(0)$ a bit more carefully.

2.3. Solution to Laplace's Equation on the Annulus. Suppose $D = B_\rho(0)$, similar to the middle panel in Figure 2.3 above, with $\rho < 1$. Our goal is to determine ρ using impedance imaging. An easy way to do this is to solve Laplace's equation with boundary conditions (2.5)/(2.8) explicitly, to see that the value of ρ is in fact encoded in the Neumann data $\gamma\nabla u \cdot \mathbf{n}$ on $\partial\Omega$. The solution to Laplace's equation can be obtained with a standard separation of variables in polar coordinates, which we carry out below. For more information on solving partial differential equations via separation of variables the interested reader can consult [24].

In what follows we will assume $\gamma = 1$, though this is merely for convenience.

The domain $\Omega \setminus D$ is an annulus, so it's convenient to write Laplace's equation (2.4) in polar coordinates

$$\frac{\partial u^2}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{2.9}$$

where $u = u(r, \theta)$ is the potential in $\Omega \setminus D$. By using (2.9) it's straightforward to check that the functions $1, \ln(r)$, and $r^{|k|}e^{ik\theta}, r^{-|k|}e^{ik\theta}$ for $k \in \mathbb{Z}$ are harmonic for $r > 0$, and hence on the annulus $\Omega \setminus D$. We will construct the relevant solution $u(r, \theta)$ as a superposition of these functions,

$$u(r, \theta) = c_0 + d_0 \ln(r) + \sum_{k \in \mathbb{Z} \setminus \{0\}} (c_k r^{|k|} + d_k r^{-|k|}) e^{ik\theta} \tag{2.10}$$

by choosing the c_k and d_k correctly.

The Dirichlet boundary condition $u = f$ on $\partial\Omega$ means $u(1, \theta) = f(\theta)$, that is,

$$c_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}} (c_k + d_k) e^{ik\theta} = f(\theta) \text{ for } \theta \in [0, 2\pi). \tag{2.11}$$

This looks like the Fourier series of the Dirichlet data f. We assume f is well-behaved, e.g., continuous and piecewise differentiable, so that the Fourier series converges pointwise to f. We can expand f in a Fourier series as

$$f(\theta) = \sum_{k \in \mathbb{Z}} f_k e^{ik\theta}, \text{ where } f_k = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) e^{-ik\theta}\, d\theta.$$

By matching the f_k with the corresponding terms on the left in (2.11) we conclude

$$c_0 = f_0 \quad \text{and} \quad c_k + d_k = f_k \text{ for } k \in \mathbb{Z} \setminus \{0\}. \tag{2.12}$$

To complete the computation we make use of the Neumann boundary condition (2.8), which takes the form $\frac{\partial u}{\partial r} = 0$ on ∂D (using the fact that the vector field $\mathbf{n}$ on ∂D points radially toward the origin, so $\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial r}$ along this inner boundary). Formally taking a term-by-term derivative of (2.10) with respect to r and then evaluating at $r = \rho$ leads to

$$\frac{d_0}{\rho} + \sum_{k \in \mathbb{Z} \setminus \{0\}} |k|(c_k \rho^{|k|-1} - d_k \rho^{-|k|-1}) e^{ik\theta} = 0 \text{ for } \theta \in [0, 2\pi). \tag{2.13}$$

Equation (2.13) can be interpreted as the Fourier series for the zero function, whose Fourier coefficients all equal zero, so we conclude that

$$d_0 = 0 \quad \text{and} \quad |k|(c_k \rho^{|k|-1} - d_k \rho^{-|k|-1}) = 0 \text{ for } k \in \mathbb{Z} \setminus \{0\}. \tag{2.14}$$

Solve (2.12) and (2.14) for c_k and d_k and substitute into (2.10) to yield the solution to Laplace's equation on the annulus satisfying the boundary conditions (2.5) and (2.8):

$$u(r, \theta) = \sum_{k \in \mathbb{Z}} \left(\frac{f_k}{1 + \rho^{2|k|}} r^{|k|} e^{ik\theta} + \frac{\rho^{2|k|} f_k}{1 + \rho^{2|k|}} r^{-|k|} e^{ik\theta} \right), \tag{2.15}$$

for all $r \in [\rho, 1]$ and $\theta \in [0, 2\pi)$; u is undefined inside D. From (2.15) we can easily compute the Neumann data on $\partial\Omega$, noting that $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial r}$ on this outer boundary, where $r = 1$:

$$\frac{\partial u}{\partial \mathbf{n}}(1, \theta) = \sum_{k \in \mathbb{Z}} \frac{|k|(1 - \rho^{2|k|})}{1 + \rho^{2|k|}} f_k e^{ik\theta}. \tag{2.16}$$

We can compute the solution to Laplace's equation on the open disk (without the void D) by using the same procedure, but omitting the $\ln(r)$ and $r^{-|k|}e^{ik\theta}$ terms in (2.10). As one might expect, the solution turns out to be exactly what one obtains from (2.15) with $\rho = 0$. The same observation holds for the Neumann data in (2.16). We should remark that we need to assume f is smooth enough so that the Fourier series (2.16) converges meaningfully, say pointwise to some continuous function.

Exercise 2.7. *Determine the potential $u(r, \theta)$ in an annulus $\rho \le r \le 1$ that satisfies $u(1, \theta) = \cos\theta$ and $\frac{\partial u}{\partial \mathbf{n}}(\rho, \theta) = 0$. Calculate $\frac{\partial u}{\partial \mathbf{n}}(1, \theta)$ to see how the radius ρ of the hole is encoded in this surface information.*

2.4. Bad Cloaking. As remarked above, one way we might try to hide an object inside Ω is to excavate a void $D = B_\rho(0)$ for some suitable $0 < \rho < 1$ and place the object inside, thereby isolating it electrically from $\partial\Omega$. The observer can gather no information concerning the object, since the Neumann data is given by (2.16) and does not depend on what is inside D. Unfortunately, the expression in (2.16) shows that the Neumann data on the right is clearly dependent on ρ. If $\rho > 0$ the observer will likely be aware that SOMETHING suspicious is going on.

To quantify this, let u_0 denote the solution to Laplace's equation on Ω with Dirichlet data $u_0 = f$, when no void D is present (Ω is empty). Let u be the solution on $\Omega \setminus D$ with $D = B_\rho(0)$, $u = f$ on $\partial\Omega$, and the boundary condition (2.8). We want to measure just how much the Neumann data for u and u_0 differ, in terms of ρ. The difference in the Neumann data for u and u_0 is, from (2.16)

$$\frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1, \theta) = -\sum_{k \in \mathbb{Z}} \frac{|k|\rho^{2|k|}}{1 + \rho^{2|k|}} f_k e^{ik\theta}. \tag{2.17}$$

A convenient way to measure the magnitude of the difference is to take the $L^2(\partial\Omega)$ norm

$$\begin{aligned} \left\| \frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1, \theta) \right\|^2_{L^2(\partial\Omega)} &:= \int_0^{2\pi} \left| \frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1, \theta) \right|^2 d\theta \\ &= \sum_{k \in \mathbb{Z}} \frac{k^2 \rho^{4|k|}}{(1 + \rho^{2|k|})^2} |f_k|^2 \end{aligned} \tag{2.18}$$

where the last line follows from equation (2.17) and Parseval's identity (see p. 133 of [24]). From (2.18) and the fact that $\frac{\rho^{4|k|}}{(1+\rho^{2|k|})^2} < \rho^4$ if $0 < \rho < 1$ and $|k| \ge 1$, we see that

$$\left\| \frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1, \theta) \right\|^2_{L^2(\partial\Omega)} \le \rho^4 \sum_{k \in \mathbb{Z}} k^2 |f_k|^2 = \rho^4 \left\| \frac{\partial u_0}{\partial \mathbf{n}}(1, \theta) \right\|^2_{L^2(\partial\Omega)}.$$

The equality on the right above follows from taking $\rho = 0$ in (2.16) and using Parseval's identity. Taking the square root of each expression above leads to the bound

$$\left\| \frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1, \theta) \right\|_{L^2(\partial\Omega)} \le \rho^2 \left\| \frac{\partial u_0}{\partial \mathbf{n}}(1, \theta) \right\|_{L^2(\partial\Omega)}. \tag{2.19}$$

In short, if the hole is small, the difference in the Neumann data will be small, proportional to ρ^2 (i.e., to the area of the hole). If the observer measures the Neumann data to finite precision, we can hide the object by making ρ so small that it perturbs the Neumann data at a level below the precision threshold—but only if the object fits! If the observer makes measurements of the Neumann data at sufficiently high precision then (2.19) will dictate a value for ρ too small to hide our object, and this approach won't work.

EXERCISE 2.8. *Calculate* $\left\|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta)\right\|_{L^2(\partial\Omega)}$ *given Dirichlet data* $f(\theta) = \cos\theta$ *(see Exercise 2.7 in the previous section).*

EXERCISE 2.9. *(A generalization of Exercise 2.7.) Show that if the Fourier coefficient* f_1 *is non-zero then we can (in principle) determine* ρ *from the boundary data, by evaluating the integral*

$$I = \int_0^{2\pi} \frac{\partial u}{\partial \mathbf{n}}(1,\theta)e^{-i\theta}\, d\theta$$

and then solving $(1-\rho^2)/(1+\rho^2) = \frac{I}{2\pi f_1}$ *for* ρ *(note* f_1 *can be determined from the Dirichlet data.) Hint: use (2.16) and orthogonality of the functions* $e^{ik\theta}$ *on* $[0, 2\pi)$.

EXERCISE 2.10. *Show that if the Fourier coefficient* f_1 *is non-zero (and note that* $f_{-1} = \overline{f_1}$ *if* f *is real) then*

$$\left\|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta)\right\|_{L^2(\partial\Omega)} \geq \frac{\sqrt{2}\rho^2|f_1|}{1+\rho^2} \geq \frac{\sqrt{2}}{2}|f_1|\rho^2$$

for $\rho \leq 1$. *Thus the Neumann data MUST differ by at least an amount proportional to* ρ^2. *Hint: simply discard all but the* $k = 1$ *and* $k = -1$ *terms in (2.18).*

3. Constructing the Cloak. What we need is a way to put a large hole in Ω, but make it look like a very small hole to an outside observer, or like no hole at all! We'll show how to do this in the case $D = B_{1/2}(0)$, though it works for a hole of any radius less than 1. The key is to surround the hole D with a ring of material that has a suitable anisotropic conductivity. The required properties of this anisotropic conductivity can be deduced from a simple change-of-variables argument.

3.1. A Change of Variables. Let's use Ω_ρ to denote the open annulus $\Omega\setminus\overline{B_\rho(0)}$ (the overline denotes the closure of the ball). Choose $\rho \in (0, 1/2)$ and let u be a twice-continuously differentiable solution to Laplace's equation on Ω_ρ, with Dirichlet data f on $\partial\Omega$ and insulating boundary condition (2.8). Let ϕ be an invertible map from $\overline{\Omega_\rho}$ to $\overline{\Omega_{1/2}}$, and suppose ϕ and ϕ^{-1} are twice-continuously differentiable. We'll use $\mathbf{x} = (x_1, x_2)$ to denote rectangular coordinates on Ω_ρ, and $\mathbf{y} = (y_1, y_2)$ for rectangular coordinates on $\Omega_{1/2}$, so $\mathbf{y} = \phi(\mathbf{x})$. Assume that ϕ maps the inner boundary $\|\mathbf{x}\| = \rho$ of Ω_ρ to the inner boundary $\|\mathbf{y}\| = 1/2$ for $\Omega_{1/2}$, ϕ maps $\|\mathbf{x}\| = 1$ to $\|\mathbf{y}\| = 1$, and that the derivative of ϕ,

$$D\phi(\mathbf{x}) = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{bmatrix}$$

is nonsingular on $\overline{\Omega_\rho}$.

Define a function v on $\Omega_{1/2}$ by $v(\mathbf{y}) = u(\phi^{-1}(\mathbf{y}))$ or equivalently, $v(\phi(\mathbf{x})) = u(\mathbf{x})$. That is, v is simply the function u "pushed forward" from Ω_ρ onto the domain $\Omega_{1/2}$ by the mapping ϕ. Because $\triangle u = 0$ in Ω_ρ, v satisfies a certain differential equation in $\Omega_{1/2}$, the focus of the following lemma.

LEMMA 3.1. *Under the assumptions above the function $v(\mathbf{y})$ satisfies the partial differential equation*

$$\nabla \cdot \sigma(\mathbf{y}) \nabla v = 0 \tag{3.1}$$

in $\Omega_{1/2}$, where $\sigma(\mathbf{y})$ denotes the 2×2 matrix

$$\sigma(\mathbf{y}) = \frac{D\phi(\mathbf{x})(D\phi(\mathbf{x}))^T}{|det(D\phi(\mathbf{x}))|} \tag{3.2}$$

evaluated at $\mathbf{x} = \phi^{-1}(\mathbf{y})$.

Proof: The proof of this lemma can certainly be done by "brute force," that is, by applying the Laplacian in $\mathbf{x}$ to both sides of the relation $u(\mathbf{x}) = v(\phi(\mathbf{x}))$ and using the chain rule, but it's a bit of a mess. A more elegant proof is obtained by using the Divergence Theorem. First, the chain rule applied to $u(\mathbf{x}) = v(\phi(\mathbf{x}))$ yields

$$\begin{aligned}\frac{\partial u}{\partial x_1}(\mathbf{x}) &= \frac{\partial y_1}{\partial x_1}\frac{\partial v}{\partial y_1}(\phi(\mathbf{x})) + \frac{\partial y_2}{\partial x_1}\frac{\partial v}{\partial y_2}(\phi(\mathbf{x}))\\ \frac{\partial u}{\partial x_2}(\mathbf{x}) &= \frac{\partial y_1}{\partial x_2}\frac{\partial v}{\partial y_1}(\phi(\mathbf{x})) + \frac{\partial y_2}{\partial x_2}\frac{\partial v}{\partial y_2}(\phi(\mathbf{x})).\end{aligned}$$

These equations can be written more compactly as $\nabla_{\mathbf{x}} u(\mathbf{x}) = (D\phi(\mathbf{x}))^T \nabla_{\mathbf{y}} v(\phi(\mathbf{x}))$, where $D\phi$ is as defined above, $\nabla_{\mathbf{x}}$ refers to the gradient in (x_1, x_2), and $\nabla_{\mathbf{y}}$ refers to the gradient in (y_1, y_2).

Let $w(\mathbf{x})$ be any continuously differentiable function defined on $\overline{\Omega_\rho}$ with $w = 0$ on $\partial\Omega_\rho$, and define $\tilde{w}$ on $\Omega_{1/2}$ via $\tilde{w}(\mathbf{y}) = w(\phi^{-1}(\mathbf{y}))$ (or $w(\mathbf{x}) = \tilde{w}(\phi(\mathbf{x}))$). Computations like those above show that $\nabla_{\mathbf{x}} w(\mathbf{x}) = (D\phi(\mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{w}(\phi(\mathbf{x}))$. Since $\triangle_{\mathbf{x}} u = 0$ in Ω_ρ ($\triangle_{\mathbf{x}}$ is the Laplacian in the $\mathbf{x}$ coordinates) we have

$$\int_{\Omega_\rho} w(\mathbf{x}) \triangle_{\mathbf{x}} u(\mathbf{x})\, d\mathbf{x} = 0. \tag{3.3}$$

Note that $w \triangle_{\mathbf{x}} u = \nabla_{\mathbf{x}} \cdot (w \nabla_{\mathbf{x}} u) - \nabla_{\mathbf{x}} w \cdot \nabla_{\mathbf{x}} u$. Substitute this into (3.3) and apply the Divergence Theorem to the first term to obtain

$$\int_{\partial\Omega_\rho} w \nabla_{\mathbf{x}} u \cdot \mathbf{n}\, ds - \int_{\Omega_\rho} \nabla_{\mathbf{x}} w \cdot \nabla_{\mathbf{x}} u\, d\mathbf{x} = 0.$$

Because $w \equiv 0$ on $\partial\Omega_\rho$ the first integral above is zero, and we obtain

$$\int_{\Omega_\rho} (\nabla_{\mathbf{x}} w)^T \nabla_{\mathbf{x}} u\, d\mathbf{x} = 0$$

since $\nabla_{\mathbf{x}} w \cdot \nabla_{\mathbf{x}} u = (\nabla_{\mathbf{x}} w)^T \nabla_{\mathbf{x}} u$. By making use of $\nabla_{\mathbf{x}} u(\mathbf{x}) = (D\phi(\mathbf{x}))^T \nabla_{\mathbf{y}} v(\phi(\mathbf{x}))$ and $\nabla_{\mathbf{x}} w(\mathbf{x}) = (D\phi(\mathbf{x}))^T \nabla_{\mathbf{y}} \tilde{w}(\phi(\mathbf{x}))$, we can write the last equation as

$$\int_{\Omega_\rho} \nabla_{\mathbf{y}} \tilde{w}(\phi(\mathbf{x}))^T (D\phi(\mathbf{x}))(D\phi(\mathbf{x}))^T \nabla_{\mathbf{y}} v(\phi(\mathbf{x}))\, d\mathbf{x} = 0.$$

Now make a change of variables to the $\mathbf{y}$ coordinate system, with $\phi(\mathbf{x}) = \mathbf{y}$ and $d\mathbf{x} = d\mathbf{y}/|\det(D\phi)|$. We find

$$\int_{\Omega_{1/2}} \left(\nabla_{\mathbf{y}} \tilde{w}(\mathbf{y})\right)^T \left(\sigma(\mathbf{y}) \nabla_{\mathbf{y}} v(\mathbf{y})\right) d\mathbf{y} = 0 \tag{3.4}$$

with $\sigma(\mathbf{y})$ as in the statement of the lemma. A straightforward calculation shows that

$$\begin{aligned}(\nabla_{\mathbf{y}}\tilde{w}(\mathbf{y}))^T(\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y}))&\\ &= \nabla_{\mathbf{y}}\tilde{w}(\mathbf{y})\cdot(\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y}))\\ &= \nabla_{\mathbf{y}}\cdot(\tilde{w}(\mathbf{y})\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y})) - \tilde{w}(\mathbf{y})\nabla_{\mathbf{y}}\cdot(\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y}))\end{aligned} \tag{3.5}$$

If we use (3.5) to replace the integrand on the left in equation (3.4) we find

$$\int_{\Omega_{1/2}} \nabla_{\mathbf{y}}\cdot(\tilde{w}(\mathbf{y})\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y}))\,d\mathbf{y} - \int_{\Omega_{1/2}} \tilde{w}(\mathbf{y})\nabla_{\mathbf{y}}\cdot(\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y}))\,d\mathbf{y} = 0.$$

An application of the Divergence Theorem to the first integral on the left above, along with the fact that $\tilde{w}(\mathbf{y}) \equiv 0$ on $\partial\Omega_{1/2}$, shows that in fact this integral equals zero, and we are left with (after dropping the leading minus sign)

$$\int_{\Omega_{1/2}} \tilde{w}(\mathbf{y})\nabla_{\mathbf{y}}\cdot(\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y}))\,d\mathbf{y} = 0. \tag{3.6}$$

The function $\tilde{w}(\mathbf{y})$ is arbitrary (since given any $\tilde{w}$ we could have chosen $w(\mathbf{x}) = \tilde{w}(\phi^{-1}(\mathbf{x}))$ back on Ω_ρ), so (3.6) holds for any continuously differentiable $\tilde{w}$. We claim this forces $\nabla_{\mathbf{y}}\cdot(\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y}))$ to be identically zero in $\Omega_{1/2}$.

To show this, let $h(\mathbf{y})$ denote the quantity $\nabla_{\mathbf{y}}\cdot(\sigma(\mathbf{y})\nabla_{\mathbf{y}}v(\mathbf{y}))$ in the integrand in (3.6). From the assumptions on ϕ and u the function h is continuous in $\Omega_{1/2}$. Suppose in contradiction to the claim that h is not identically zero on $\Omega_{1/2}$, say $h(\mathbf{y}) > 0$ at some point $\mathbf{y}_0$. Since $h(\mathbf{y})$ continuous we have $h(\mathbf{y}) > 0$ in a ball $B_\delta(\mathbf{y}_0) \subset \Omega_{1/2}$ for some $\delta > 0$. We can choose some function $\tilde{w}(\mathbf{y}) \geq 0$ which is positive in $B_\delta(\mathbf{y}_0)$ and $\tilde{w}(\mathbf{y}) \equiv 0$ outside of $B_\delta(\mathbf{y}_0)$. As a result the product $\tilde{w}(\mathbf{y})h(\mathbf{y}) \geq 0$ in $\Omega_{1/2}$ and $\tilde{w}(\mathbf{y})h(\mathbf{y})$ is not identically zero. But then the integral in (3.6) cannot equal zero, a contradiction. We conclude that $h(\mathbf{y}) = \nabla\cdot(\sigma(\mathbf{y})\nabla v(\mathbf{y}))) = 0$ in $\Omega_{1/2}$, and this proves the lemma.□

The matrix σ defined by (3.2) is positive definite (see Exercise 3.2 below). Comparison of equation (3.1) to (2.7) shows that v can be considered as the electric potential inside $\Omega_{1/2}$ corresponding to the anisotropic conductivity σ. It is this observation that will allow us to design an anisotropic conductivity to cloak the ball $B_{1/2}(0)$.

EXERCISE 3.2. *Show that the matrix $\sigma(\mathbf{y})$ defined by equation (3.2) is symmetric and positive definite for each $\mathbf{y}$, that is, satisfies $\mathbf{w}^T\sigma(\mathbf{y})\mathbf{w} > 0$ for each non-zero vector $\mathbf{w} \in \mathbb{R}^2$.*

3.2. Designing the Cloak. The properties we need from the layer of anisotropic material surrounding $D = B_{1/2}(0)$ can be deduced by considering functions $\phi : \Omega_\rho \to \Omega_{1/2}$ with the specific form

$$\phi(\mathbf{x}) = \frac{\psi(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\mathbf{x}, \tag{3.7}$$

so that $\mathbf{y} = \phi(\mathbf{x})$ means $y_1 = \frac{\psi(\|\mathbf{x}\|)}{\|\mathbf{x}\|}x_1$ and $y_2 = \frac{\psi(\|\mathbf{x}\|)}{\|\mathbf{x}\|}x_2$, where ψ is a function chosen so that

- $\psi(\rho) = 1/2$ (ϕ maps the inner boundary of Ω_ρ to that of $\Omega_{1/2}$);

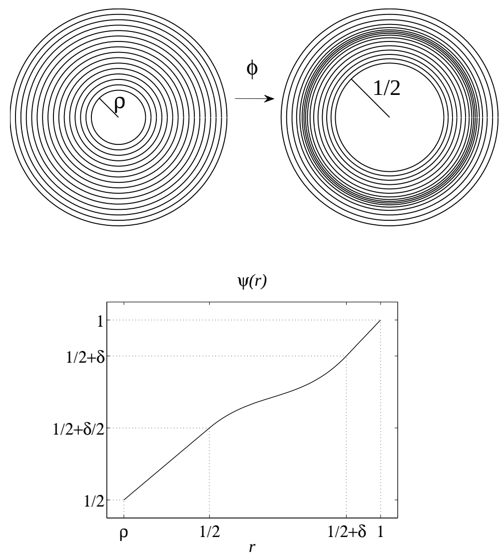

FIG. 3.1. *An example of a function* $\phi(\mathbf{x}) = \frac{\psi(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\mathbf{x}$*, where* $\psi(r)$ *is defined via (3.8). Note that* ϕ *maps a circle of radius* r *to a circle of radius* $\psi(r)$*.*

- For some $\delta \in (0, 1/2)$ we have $\psi(r) = r$ for $1/2 + \delta < r < 1$ (so ϕ fixes a neighborhood $1/2 + \delta < \|\mathbf{x}\| \leq 1$ of the outer boundary at $\|\mathbf{x}\| = 1$);
- The function ψ is twice-continuously differentiable, with $\psi'(r) \geq d_0 > 0$ for some d_0, so ψ will be strictly increasing and invertible.

The mapping ϕ simply "pushes" points in Ω_ρ radially outward from the origin, at least for $\rho \leq \|\mathbf{x}\| < 1/2 + \delta$. There are many ways to rig up such a ψ, for example,

$$\psi(r) = \begin{cases} \frac{1}{2} + \frac{\delta}{1-2\rho}(r - \rho), & \rho \leq r \leq \frac{1}{2} \\ g(r) & \frac{1}{2} < r < \frac{1}{2} + \delta \\ r & \frac{1}{2} + \delta \leq r \leq 1 \end{cases} \tag{3.8}$$

where $g(r)$ is a suitably chosen function to smoothly interpolate between the two regions on which ψ is linear. The precise formula for g isn't important at the moment. A typical ψ and the resulting mapping of Ω_ρ to $\Omega_{1/2}$ is shown in Figure 3.1.

Under such a mapping ϕ we have $\mathbf{y} = \mathbf{x}$ in a neighborhood $1/2 + \delta \leq r \leq 1$ of the outer boundary, and so $u \equiv v$ in this region. The function $v = u \circ \phi^{-1}$ also has zero Neumann data on the inner boundary $\|\mathbf{y}\| = 1/2$. Specifically, we have

$$\begin{aligned} \frac{\partial v}{\partial \mathbf{n}}\Big|_{\|\mathbf{y}\|=1/2} &= -\frac{\partial v}{\partial \|\mathbf{y}\|}\Big|_{\|\mathbf{y}\|=1/2} \ \left(\text{recall } \frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial \|\mathbf{y}\|} \text{ on } \|\mathbf{y}\| = 1/2\right) \\ &= -\frac{\partial \|\mathbf{x}\|}{\partial \|\mathbf{y}\|}\frac{\partial u}{\partial \|\mathbf{x}\|}\Big|_{\|\mathbf{x}\|=\rho} \\ &= -\frac{1-2\rho}{\delta}\frac{\partial u}{\partial \|\mathbf{x}\|}\Big|_{\|\mathbf{x}\|=\rho} \\ &= 0 \end{aligned}$$

where we make use of $\|\mathbf{y}\| = \psi(\|\mathbf{x}\|)$ and the first case in (3.8), which yields $\frac{\partial \|\mathbf{y}\|}{\partial \|\mathbf{x}\|} = \delta/(1 - 2\rho)$ at $\|\mathbf{x}\| = \rho$, hence $\frac{\partial \|\mathbf{x}\|}{\partial \|\mathbf{y}\|} = (1 - 2\rho)/\delta$.

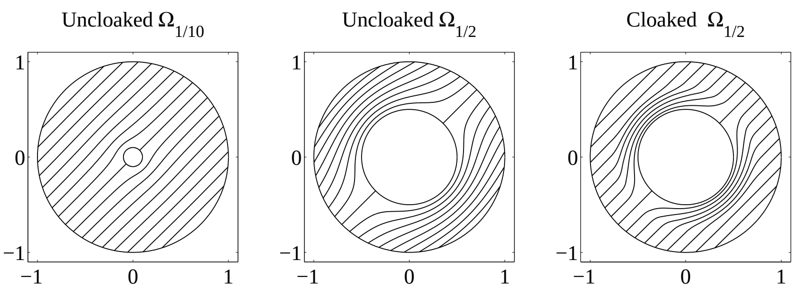

FIG. 3.2. *Comparison of uncloaked and cloaked solutions on annuli. The left and middle graphs show flow lines of the current* $\mathbf{J} = -\gamma\nabla u$*, where* u *is a solution of Laplace's equation with Dirichlet condition* $f(\theta) = \cos\theta + \sin\theta$ *(the potential applied by the observer) on the outer boundary of annuli with constant conductivity. The graph on the right shows flow lines of the current* $\mathbf{J} = -\sigma\nabla v$ *for the approximately cloaked ball, with anisotropic conductivity* σ *corresponding to* $\rho = 1/10$.

EXERCISE 3.3. *Write out the conditions on* $g(r), g'(r)$*, and* $g''(r)$ *at* $r = 1/2$ *and* $r = 1/2 + \delta$ *that make* ψ *in (3.8) twice-continuously differentiable. In the case* $\rho = 1/10, \delta = 1/10$*, find such a function* g*. Hint: try a 5th degree polynomial; a computer algebra system might help!*

3.3. The Conductivity σ is an Approximate Cloak. We claim that the conductivity σ defined by (3.2) can be used to cloak the void D to any desired degree, with ρ as a parameter that controls the quality of the cloak. To see this, note that the matrix σ corresponds to the scalar conductivity 1 on $\Omega_{1/2}$ when $\|\mathbf{y}\| > 1/2 + \delta$, that is, in a neighborhood of the outer boundary, and as remarked above v and u are equal in this region. This means that u and v have precisely the same Dirichlet and Neumann data on $\partial\Omega$. In the "cloaking region" $1/2 < \|\mathbf{y}\| < 1/2 + \delta$ the quantity $\sigma(y)$ corresponds to an anisotropic conductivity. In light of the estimate (2.19) and $\partial v/\partial\mathbf{n} = \partial u/\partial\mathbf{n}$ on $\partial\Omega$ we see that

$$\left\| \frac{\partial v}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta) \right\|_{L^2(\partial\Omega)} = \rho^2 \left\| \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta) \right\|_{L^2(\partial\Omega)} \tag{3.9}$$

even though v is the potential on a region $\Omega_{1/2}$ with a central hole of radius 1/2. By making ρ close to zero we can make the Neumann data for v as close as we like to the Neumann data for u_0—we can make the region with a hole of size 1/2 look as close to empty as we like! See Figure 3.2 for an example.

3.4. Behavior in the Cloaking Region. It's extremely interesting to examine the behavior of σ in the inner cloaking region $1/2 \le \|\mathbf{y}\| \le 1/2 + \delta/2$, near $\|\mathbf{y}\| = 1/2$ (this region corresponds to $\rho < \|\mathbf{x}\| < 1/2$, the first case for ψ in (3.8).) In particular, let's examine the eigenvectors and eigenvalues of σ, corresponding the directions of maximal and minimal conductivity.

From equation (3.7) it's not hard to compute that

$$D\phi = (\psi'(r)/r^2 - \psi(r)/r^3) \begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix} + (\psi(r)/r)\mathbf{I} \tag{3.10}$$

where $\mathbf{I}$ is the identity matrix and $r = \|\mathbf{x}\| = \psi^{-1}(\|\mathbf{y}\|)$. In particular, note that $D\phi$ is symmetric, so that from equation (3.2) we have $\sigma = (D\phi)^2/|\det(D\phi)|$.

EXERCISE 3.4. *Let* $\mathbf{v}$ *be an eigenvector with eigenvalue* μ *for an* $n \times n$ *matrix* $\mathbf{A}$*, and let* $\mathbf{B} = \mathbf{A}^2/|det(A)|$*. Show that* $\mathbf{v}$ *is also an eigenvector for* $\mathbf{B}$*, with eigenvalue* $\lambda = \mu^2/|det(\mathbf{A})|$*.*

EXERCISE 3.5. *Show that the* 2×2 *matrix with entries* x_1^2, x_1x_2, x_2^2 *in equation (3.10) has (orthogonal) eigenvectors* $[x_1, x_2]^T$ *and* $[-x_2, x_1]^T$*, with eigenvalues* $r^2 = \|\mathbf{x}\|^2$ *and* 0*, respectively.*

If we can compute the eigenvectors and eigenvalues for $D\phi$ then we can make use of Exercise 3.4 to find these quantities for σ. The eigenvectors and eigenvalues for $D\phi$ follows easily from Exercise 3.5: the multiplication of the matrix with entries x_1^2, etc. by $(\psi'(r)/r^2 - \psi(r)/r^3)$ and the resulting shift by $(\psi(r)/r)\mathbf{I}$ shows that $D\phi$ also has eigenvectors $\mathbf{v}_1 = [x_1, x_2]^T$ and $\mathbf{v}_2 = [-x_2, x_1]^T$, with corresponding eigenvalues $\mu_1 = (\psi'(r) - \psi(r)/r) + \psi(r)/r = \psi'(r)$ and $\mu_2 = \psi(r)/r$. Note also that $\det(D\phi) = \mu_1\mu_2$. From Exercise 3.4 we then find that σ has eigenvectors $\mathbf{v}_m = \mathbf{v}_1$ and $\mathbf{v}_M = \mathbf{v}_2$ (the same as $D\phi$, but relabeled to indicate what will be the directions of maximum and minimum conductivity)). The corresponding eigenvalues/conductivities are

$$\begin{aligned}\gamma_m &= \frac{\mu_1^2}{\mu_1\mu_2} = \frac{r\psi'(r)}{\psi(r)}\\ \gamma_M &= \frac{\mu_2^2}{\mu_1\mu_2} = \frac{\psi(r)}{r\psi'(r)}.\end{aligned} \tag{3.11}$$

In particular, the conductivities are reciprocals of each other!

The vector $\mathbf{v}_m$ points radially outward from the origin and $\mathbf{v}_M$ is tangential to any circle centered at the origin. Indeed, at a point $x_1 = r\cos(\theta), x_2 = r\sin(\theta)$ we may as well take $\mathbf{v}_m = (\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath}$, since eigenvectors can be rescaled. Similarly, we may take $\mathbf{v}_M = -(\sin\theta)\hat{\imath} + (\cos\theta)\hat{\jmath}$. If we use (3.11) to examine the behavior of σ in the inner cloaking region $1/2 < \|\mathbf{y}\| < 1/2 + \delta/2$ (corresponding to $\rho < \|\mathbf{x}\| < 1/2$) and make use of (3.8) we find the conductivities in this region are given by

$$\gamma_m(r) = \frac{2r\delta}{1 + 2r\delta - 2\rho - 2\delta\rho}, \quad \gamma_M(r) = \frac{1 + 2r\delta - 2\rho - 2\delta\rho}{2r\delta} = \frac{1}{\gamma_m(r)}. \tag{3.12}$$

Note that we can express the eigenvalues in terms of $\mathbf{y}$ via $r = \psi^{-1}(\|\mathbf{y}\|)$. At the inner surface $\|\mathbf{y}\| = 1/2$ (corresponding to $r = \rho$) on $\Omega_{1/2}$ we have

$$\gamma_m = \frac{2\rho\delta}{1 - 2\rho}, \quad \gamma_M = \frac{1 - 2\rho}{2\rho\delta}.$$

When ρ is close to zero, $\gamma_M \approx \frac{1}{2\rho\delta}$ is large, so the conductivity in the tangential direction on the circle $\|\mathbf{y}\| = 1/2$ is very large. Similarly, $\gamma_m \approx 2\rho\delta$ is close to zero in this case, so the conductivity in the normal direction is low. Physically, a "ray" (really, an electron) approaches $\|\mathbf{y}\| = 1/2$ and is diverted in the direction of the high conductivity, routed tangentially around the ball $B_{1/2}(0)$, then ejected out the other side to continue on its way. Mr. Spock's "selective bending of light rays" (or in this case, electric current) is realized, but now grounded in the real laws of physics! For example, look at the flows near $R = 1/2$ in the rightmost graph in Figure 3.2.

EXERCISE 3.6. *Work out the eigenvalues for* σ *in the transition region* $1/2 + \delta/2 < \|\mathbf{y}\| < 1/2 + \delta$ *(corresponding to* $1/2 < \|\mathbf{x}\| < 1/2 + \delta$*) in terms of* $g(r)$ *and* $g'(r)$*. Show that the conductivities smoothly transition from those in (3.12) to those for an isotropic conductor of conductivity* 1*.*

EXERCISE 3.7. *Carry out the analogous computations in three-dimensions! (It really is quite the same: Lemma 3.1 still holds, and the remaining computations are similar to the 2D case. You don't need to solve Laplace's equation.)*

3.5. The Perfect Cloak. Of course, it's natural to consider letting $\rho \to 0^+$ above, to obtain the "perfect invisibility cloak." This can be done! (See section 4 of [10] for how to rigorously carry out a singular change of variables to yield a perfect cloak.) However, if we look at the eigenvalues for σ, we see that γ_m evaluated along the inner boundary $\|\mathbf{y}\| = 1/2$ goes to zero as $\rho \to 0$, while γ_M goes to infinity; both eigenvectors are unchanged. This corresponds to perfect conductance around $\|\mathbf{y}\| = 1/2$, perfect insulation across this curve, which may not be physically realistic. Still, by making ρ small but non-zero we can get a "practical" cloak of any desired strength without singular behavior.

EXERCISE 3.8. *(This is a bit more involved!) Suppose the region $D = B_{1/2}(0)$ is not a void, but a region of (say) homogeneous isotropic conductivity α (but otherwise no object stuck inside.) Adapt the procedure above to show how to cloak D to any desired degree, so that from the outer boundary $\|\mathbf{x}\| = 1$, Ω looks like a homogeneous conductor with conductivity 1. Show that as the "cloaking quality" parameter ρ tends to zero, the current flowing through D is "pinched off" (no current flows through D).*

3.6. Anisotropic Conductors and Metamaterials. Although many natural materials have anisotropic conductivity, how does one actually design a material with desired anisotropic properties? One approach is to use homogeneous, isotropic materials and introduce periodic microstructure, e.g., put holes or cracks in the material in a specific pattern, but on a very small scale. By imposing periodic microstructure we obtain a material that, macroscopically, appears to have anisotropic properties. The mathematical theory involved in analyzing how periodic microstructure yields given macroscopic properties is called *homogenization*, and the techniques apply to far more than electrical conduction; they can be applied to many situations involving a physical system governed by differential equations.

We won't go into the details of homogenization here, but as a simple example, in [2] the authors show how one can obtain a conductive material that appears macroscopically to be an anisotropic electrical conductor, by introducing periodic cracks into a homogeneous isotropic conductor. Specifically, consider the box $-\epsilon < x_1, x_2 < \epsilon$ in $\mathbb{R}^2$, with isotropic conductivity γ. We introduce an insulating crack into the box; the crack is linear with center at $(0, 0)$, and lies at angle α with respect to horizontal. The authors in [2] show that if we "tile" a region Ω in the plane with a collection of these 2ϵ by 2ϵ boxes and let $\epsilon \to 0$, the region Ω has effective anisotropic conductivity σ of the form

$$\sigma = \gamma \mathbf{I} - \gamma R \begin{bmatrix} \sin^2(\alpha) & -\sin(\alpha)\cos(\alpha) \\ -\sin(\alpha)\cos(\alpha) & \cos^2(\alpha) \end{bmatrix}$$

where R is a parameter that depends on the angle α and the length of the crack relative to the width of the box. By adjusting the angle and length of the cracks (relative to their spacing), as well as γ, one can in principle obtain any anisotropic conductivity profile. Similar results can be obtained by introducing periodic holes or other shapes.

4. Conclusion. In this article, we have described the essential idea behind one approach to cloaking, in two dimensions for imaging with impedance tomography.

More realistically, one could apply these ideas to Maxwell's equations in three dimensions (see p. 358-361 of [24] for an overview of Maxwell's equations), at nonzero frequencies, and use a singular change of variables in order to achieve a perfect cloak (rather than a near-cloak), as derived in [6]. One key question of interest is whether one can cloak over a large range of frequencies, rather than merely at a particular frequency, as the range of frequencies is severely restricted for some cloaking formulations [3]. However, by avoiding metamaterials whose properties depend on resonance, researchers have recently discovered that cloaking for a range of frequencies in the electromagnetic spectrum may indeed be possible, and may even work for visible light [14, 15].

The field of cloaking is extremely active, with many intriguing ideas emerging. For example, Lai and colleagues have designed a device that can cloak an object from a distance (the device is designed specially for a particular object at a specified location relative to the cloaking device) [12]. Cloaking effects can also be generated by anomalous localized resonance [18], which occurs near a "superlens," a metamaterial with negative refraction index that can yield resolution finer than the wavelength of the light being used to generate the image [20]. Cloaking has been explored in contexts other than electromagnetic waves, such as for elasticity waves [17] and for matter waves (quantum cloaking) [26].

The topic of cloaking suggests many interesting research projects for undergraduates to pursue. Here are a few open-ended suggestions for possible directions to explore (no claims are made or implied concerning the ease or even possibility of solving these!):

1. Our approach to cloaking was to make a large hole look like a small hole. Can we do the reverse—make a small hole look large? Even more generally, can this change-of-variables technique be used to disguise, rather than hide D? For example, can $D = B_{1/2}(0)$ be made to look like an ellipse or some other shape? What are the limitations?
2. Could one construct a "directional cloak" that renders an object (approximately) invisible from some directions, fully visible from others? Think of some kind of device you carry into battle, so that from the front (where your enemies are) you're invisible, but from behind (where your allies are) you're visible.
3. Another form of energy that has been used for imaging is heat. Suppose $v(x_1, x_2, t)$ satisfies the heat equation $v_t - \triangle v = 0$ in the unit disk $\Omega = B_1(0)$ (here v is the temperature of Ω). For simplicity, suppose v is time-harmonic, that is, $v(x_1, x_2, t) = e^{i\omega t} u(x_1, x_2)$. Then $\triangle u + i\omega u = 0$. An observer probes the interior of Ω by imposing a temperature $u = f$ on $\partial\Omega$, then measuring the heat flux $\frac{\partial u}{\partial \mathbf{n}}$ on $\partial\Omega$. Can we cloak a void $D = B_{1/2}(0)$ using the technique for impedance imaging? If $\omega = 0$ it's the same problem, so assume $\omega > 0$.

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Cloaking Research

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July 2017

1 Exercises

An Enumeration of the Exercises in Kurt Bryan and Tanya Leise's paper *Impedance Imaging, Inverse Problems, and Harry Potter's Cloak*
Exercise 2.1:
First we must verify that the function $u(x_1, x_2)$ satisfies the Laplacian $\Delta u = 0$ coupled with the Dirichlet data $u(x_1, x_2) = f(\theta)$ on $\partial\Omega$. Since $\dfrac{\partial u}{\partial x_1} = a$ and $\dfrac{\partial u}{\partial x_2} = b$, we have $\Delta u = \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0$. In verifying the Dirichlet data, we find $ax_1 + bx_2 + c = a\cos\theta + b\sin\theta + c$ and it is clear that the parametrization $x_1 = \cos\theta$ and $x_2 = \sin\theta$ yields the desired $u(x_1, x_2) = f(\theta)$
Exercise 2.2:
An isotropic conductor has no directional properties. We seek to find an orthogonal basis of vectors that span $\mathbb{R}^2$ of magnitude γ. The simplest way of doing this is simply multiplying γ to the two unit vectors $\vec{e_1}$ and $\vec{e_2}$, yielding

$$\vec{v_1} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ \gamma \end{bmatrix}$$

To check that $\vec{v_1}$ and $\vec{v_2}$ satisfy the conditions, we verify the following:

1. $\vec{v_1} \times \vec{v_2} = \gamma \cdot 0 + 0 \cdot \gamma = 0$
2. $\sqrt{\vec{v_1} \cdot \vec{v_1}} = \sqrt{\gamma^2 + 0} = \gamma$
3. $\sqrt{\vec{v_2} \cdot \vec{v_2}} = \sqrt{0 + \gamma^2} = \gamma$
4. $c_1\vec{v_1} + c_2\vec{v_2} = 0$ only has the trivial solution $c_1 = c_2 = 0$ therefore $\text{span}([\vec{v_1}, \vec{v_2}]) = \mathbb{R}^2$

Finally, we write $\sigma = [\vec{v_1} | \vec{v_2}] = \gamma I$:

$$\sigma = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}$$

Exercise 2.3:
We use $\vec{v_M}$ and $\vec{v_m}$ as the two orthonormal eigenvectors corresponding to the eigenvalues γ_M and γ_m in the decomposition of σ. Then, using $\sigma = PDP^{-1} = PDP^T$, where P is the matrix with the eigenvectors and D is the diagonal matrix with the eigenvalues of σ (in the same order as the eigenvectors in P), we have

$$\sigma = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \gamma_M & 0 \\ 0 & -\gamma_m \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Simplifying this, we obtain

$$\sigma = \frac{1}{2} \begin{bmatrix} \gamma_M + \gamma_m & \gamma_M - \gamma_m \\ \gamma_M - \gamma_m & \gamma_M + \gamma_m \end{bmatrix}$$

As $\gamma_m \to 0^+$,

$$\sigma \to \frac{1}{2} \begin{bmatrix} \gamma_M & \gamma_M \\ \gamma_M & \gamma_M \end{bmatrix}$$

Exercise 2.4:
First, we calculate $||\vec{v}||^2$. $||\vec{v}||^2 = \vec{v}\cdot\vec{v}$. We use $\vec{v} = \sum_{k=1}^n \alpha_k \vec{v}_k$ where $\vec{v}_k$ are orthonormal eigenvectors for σ and α_k are some scalars. Then, we may compute $||\vec{v}||^2 = (\sum_{i=1}^n \alpha_i \vec{v}_i)(\sum_{k=1}^n \alpha_k \vec{v}_k)$. Now, noting that the vectors $\vec{v}_k$ are orthonormal, $\vec{v}_i \cdot \vec{v}_k = 0$ if $i \neq k$ and $\vec{v}_i \cdot \vec{v}_k = 1$ if $i = k$. Thus, we have $||\vec{v}||^2 = \sum_{k=1}^n \alpha_k^2$ and since we have fixed $||\vec{v}|| = 1$, $\sum_{k=1}^n \alpha_k^2 = 1$.
Now we consider $||\sigma\vec{v}||^2$ for an arbitrary vector $\vec{v}$. $||\sigma\vec{v}||^2 = \sigma\vec{v}\cdot\sigma\vec{v}$. $\sigma\vec{v}\cdot\sigma\vec{v} = (\sum_{i=1}^n \alpha_i \sigma\vec{v}_i)(\sum_{k=1}^n \alpha_k \sigma\vec{v}_k)$. Note that the $\vec{v}_i$ and $\vec{v}_k$ are the eigenvectors of σ so we have $\sigma\vec{v}_k = \lambda_k\vec{v}_k$. Using this and the orthonormal nature of the vectors, we have $||\sigma\vec{v}||^2 = \sum_{k=1}^n \alpha_k^2\lambda_k^2$.
Finally, we must show that the previous expression is less than or equal to $||\sigma\vec{v}_n||^2$, where $\vec{v}_n$ is the eigenvector corresponding to the largest eigenvalue λ_n. Since $\sigma\vec{v}_n = \lambda_n\vec{v}_n$ and $||\vec{v}_n|| = 1$, we have $||\sigma\vec{v}_n||^2 = ||\lambda_n\vec{v}_n||^2 = \lambda_n^2$. Note that since λ_n is the largest eigenvalue, we have the inequality

$$\sum_{k=1}^n \alpha_k^2\lambda_k^2 \leq \sum_{k=1}^n \alpha_k^2\lambda_n^2 = \lambda_n^2$$

where the last equality follows from fixing $||\vec{v}|| = \sum_{k=1}^n \alpha_k^2 = 1$. Thus, we have

$$||\sigma\vec{v}||^2 \leq ||\sigma\vec{v}_n||^2$$

Example 2.5 is explained in the paper so it is omitted here.
Exercise 2.6: We must verify three conditions: the Laplacian, the Neumann data on ∂D, and the Dirichlet data on $\partial\Omega$. It is easier to verify the Laplacian in polar coordinates, using $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Then, we have

$$u(r,\theta) = (\sin\theta + \cos\theta)\Big(\frac{4r^2+1}{5r}\Big)$$

In polar coordinates, the Laplacian becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = 0$$

Computing the relevant derivatives, we have

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} = (-\sin\theta - \cos\theta)\Big(\frac{4r^2+1}{5r^3}\Big)$$

$$\frac{1}{r}\frac{\partial u}{\partial r} = (\sin\theta + \cos\theta)\Big(\frac{4r^2-1}{5r^3}\Big)$$

$$\frac{\partial^2 u}{\partial r^2} = (\sin\theta + \cos\theta)\Big(\frac{2}{5r^3}\Big)$$

Adding these terms yields the expected verification of the Laplacian.
On ∂D, we must have $\frac{\partial u}{\partial \mathbf{n}} = 0$. We use $\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial r}$. Setting $\frac{\partial u}{\partial r} = 0$, we find $r = \frac{1}{2}$ which we expect since we have removed a ball $D = B_{1/2}(0)$ (a ball of radius 1/2 centered at 0).
Finally, we check the Dirichlet data on $\partial\Omega$. To do so, we simply show that $u(1,\theta) = f(\theta) = \cos\theta + \sin\theta$. Plugging into $u(r,\theta)$, we find $u(1,\theta) = \cos\theta + \sin\theta$
Exercise 2.7:
The paper covers the general solution to the Laplacian on an annulus:

$$u(r,\theta) = \sum_{k\in\mathbb{Z}}\Big(\frac{f_k}{1+\rho^{2|k|}}r^{|k|}e^{ik\theta} + \frac{\rho^{2|k|}f_k}{1+\rho^{2|k|}}r^{-|k|}e^{ik\theta}\Big)$$

This solution already accounts for $\frac{\partial u}{\partial \mathbf{n}}(\rho,\theta) = 0$ so the only task left is to obtain f_k for $f(\theta) = \cos\theta$. For f_k, we have the expression

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\,d\theta$$

To evaluate this integral, we may use $e^{-ik\theta} = \cos k\theta - i \sin k\theta$. Then, we have two separate integrals in the expression for f_k:

$$\int_0^{2\pi} \cos\theta \cos k\theta \, d\theta$$

and

$$\int_0^{2\pi} \cos\theta \sin k\theta \, d\theta$$

The set $\{1, \sin\theta, \cos\theta, \sin 2\theta, \cos 2\theta \cdots\}$ is mutually orthogonal so the only nonzero integral occurs when $k = 1$:

$$f_1 = \frac{1}{2\pi} \int_0^{2\pi} \cos^2\theta \, d\theta = \frac{1}{2}$$

Plugging this into the expression for $u(r,\theta)$, we find

$$u(r,\theta) = \frac{e^{i\theta}}{2(1+\rho^2)} \Big(r + \frac{\rho^2}{r} \Big)$$

Exercise 2.8:
We must simply utilize the work from Exercise 2.7 showing that for the Dirichlet boundary condition $u(1,\theta) = f(\theta) = \cos\theta$, the only Fourier coefficient that is nonzero is $f_1 = \frac{1}{2}$. Using this in Equation 2.18 with $k = 1$, we have

$$\Big|\Big| \frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta) \Big|\Big|^2_{L^2\Omega} = \frac{\rho^4}{4(1+\rho^2)^2}$$

And taking the square root of both sides, we obtain

$$\Big|\Big| \frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta) \Big|\Big|_{L^2\Omega} = \frac{\rho^2}{2(1+\rho^2)}$$

Exercise 2.9:
If $f_1 \neq 0$, then we have

$$\frac{\partial u}{\partial \mathbf{n}}(1,\theta) = \frac{1-\rho^2}{1+\rho^2} f_1 e^{i\theta} + \sum_{k\in\mathbb{Z}/\{1\}} \frac{|k|(1-\rho^{2|k|})}{1+\rho^{2|k|}} f_k e^{ik\theta}$$

However, note that upon evaluating the integral expression for I, we observe that the orthogonality of the functions $e^{ik\theta}$ guarantees that

$$\int_0^{2\pi} e^{ik_1\theta} e^{-ik_2\theta} \, d\theta = \begin{cases} 0 & k_1 \neq k_2 \\ 1 & k_1 = k_2 \end{cases}$$

And thus, the only nonzero term for the integral occurs when $k = 1$:

$$I = \int_0^{2\pi} \frac{\partial u}{\partial \mathbf{n}}(1,\theta) e^{-i\theta} \, d\theta = \int_0^{2\pi} \frac{1-\rho^2}{1+\rho^2} f_1 \, d\theta$$

Evaluating this expression, we find the desired relationship

$$\frac{1-\rho^2}{1+\rho^2} = \frac{I}{2\pi f_1}$$

Exercise 2.10:
To obtain the inequality, we omit all terms except for $k = -1, 1$. Using $k = -1, 1$ in Equation 2.16, we have

$$\Big|\Big| \frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta) \Big|\Big|^2_{L^2\Omega} \geq \frac{\rho^4}{(1+\rho^2)^2} |f_{-1}|^2 + \frac{\rho^4}{(1+\rho^2)^2} |f_1|^2$$

If f is real, then $f_{-1} = \overline{f_1}$ so $|f_{-1}|^2 = |f_1|^2$ and:

$$\Big|\Big| \frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta) \Big|\Big|^2_{L^2\Omega} \geq \frac{2\rho^4}{(1+\rho^2)^2} |f_1|^2$$

Taking the square root and noting that for $\rho \leq 1$, $1+\rho^2 \leq 2$, we find the final expression

$$\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta)\Big|\Big|_{L^2\Omega} \geq \frac{\sqrt{2}}{2}|f_1|\rho^2$$

Lemma 3.1 is stated and proved in the paper so it is omitted here.
Exercise 3.2:
We have defined $\sigma(\vec{y})$ as

$$\sigma(\vec{y}) = \frac{D\phi(\vec{x})(D\phi(\vec{x}))^T}{det(D\phi(\vec{x}))}$$

Simple computation yields

$$\sigma(\vec{y}) = \frac{\begin{bmatrix} \left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2 & \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_1} + \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_1} + \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_2} & \left(\frac{\partial y_2}{\partial x_1}\right)^2 + \left(\frac{\partial y_2}{\partial x_2}\right)^2 \end{bmatrix}}{\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1}}$$

Then, we consider an arbitrary vector $\vec{w} = [w_1\, w_2]$. Computing $\vec{w}^T\sigma(\vec{y})\vec{w}$ (omitting $\det \sigma(\vec{y})$ since the matrix $\sigma(\vec{y})$ is nonsingular) yields

$$\vec{w}^T\sigma(\vec{y})\vec{w} = \Big(w_1\frac{\partial y_1}{\partial x_1} + w_2\frac{\partial y_2}{\partial x_1}\Big)^2 + \Big(w_1\frac{\partial y_1}{\partial x_2} + w_2\frac{\partial y_2}{\partial x_2}\Big)^2$$

where the presence of squares (and lack of imaginary terms!) guarantees $\vec{w}^T\sigma(\vec{y})\vec{w}$ is not negative. We must confirm that $\vec{w}^T\sigma(\vec{y})\vec{w} \neq 0$. We do so using a proof by contradiction. If $\vec{w}\sigma(\vec{y})\vec{w} = 0$, then we must have

$$w_1\frac{\partial y_1}{\partial x_1} + w_2\frac{\partial y_2}{\partial x_1} = 0$$

and

$$w_1\frac{\partial y_1}{\partial x_2} + w_2\frac{\partial y_2}{\partial x_2} = 0$$

simultaneously. However, note that the determinant of the coefficient matrix is $\det D\phi(\vec{x})$. Thus, the fact that $D\phi(\vec{x})$ is nonsingular ($\det D\phi(\vec{x}) \neq 0$) guarantees that the only solution to this system is the trivial one $w_1 = w_2 = 0$. Therefore, $\sigma(\vec{y})$ is positive definite.
Exercise 3.3:
We must construct $g(r)$ such that $\psi(r)$ is twice-continuously differentiable.

$$\psi(r) = \begin{cases} \frac{1}{2} + \frac{\delta}{1-2\rho}(r-\rho) & \rho \leq r < \frac{1}{2} \\ g(r) & \frac{1}{2} < r < \frac{1}{2}+\delta \\ r & \frac{1}{2}+\delta \leq r \leq 1 \end{cases}$$

We have several conditions that $g(r)$ must satisfy:

1. $g(\frac{1}{2}) = \frac{1}{2} + \frac{\delta}{1-2\rho}(\frac{1}{2}-\rho)$
2. $g'(\frac{1}{2}) = \frac{1}{2} + \frac{\delta}{1-2\rho}$
3. $g(\frac{1}{2}+\delta) = \frac{1}{2}+\delta$
4. $g'(\frac{1}{2}+\delta) = 1$
5. $g''(\frac{1}{2}) = 0$
6. $g''(\frac{1}{2}+\delta) = 0$

We may try a fifth degree polynomial of the form $g(r) = a_5r^5 + a_4r^4 + a_3r^3 + a_2r^2 + a_1r + a_0$ where $g'(r) = 5a_5r^4 + 4a_4r^3 + 3a_3r^2 + 2a_2r + a_1$ and $g''(r) = 20a_5r^3 + 12a_4r^2 + 6a_3r + 2a_2$. We use the previously enumerated conditions to find expressions for the constants $a_0 \cdots a_5$ (a computer algebra program may be employed here).

Exercise 3.4:

Since $\vec{v}$ is an eigenvector corresponding to the eigenvalue μ for the matrix A, we have $A\vec{v} = \mu\vec{v}$. Thus, if we define $B = A^2/|det(A)|$, we have:

$$B\vec{v} = \frac{A}{|det(A)|}A\vec{v} = \mu\frac{A\vec{v}}{|det(A)|} = \frac{\mu^2}{|det(A)|}\vec{v}$$

And $\vec{v}$ is an eigenvector for B corresponding to the eigenvalue $\lambda = \mu^2/|det(A)|$

Exercise 3.5:

Here we simply follow the steps in finding eigenvalues and eigenvectors. First, we obtain the eigenvalues:

$$A - \lambda I = \begin{bmatrix} x_1^2 - \lambda & x_1x_2 \\ x_1x_2 & x_2^2 - \lambda \end{bmatrix}$$

Then, using the characteristic equation, we find $\det(A - \lambda I) = \lambda^2 - (x_1^2 + x_2^2)\lambda = 0$ and $\lambda = 0, x_1^2 + x_2^2$. Next, we find the null space using each of these eigenvalues.

$$A - 0I = \left[\begin{array}{cc|c} x_1^2 - \lambda & x_1x_2 & 0 \\ x_1x_2 & x_2^2 - \lambda & 0 \end{array}\right]$$

Row reduction yields

$$\left[\begin{array}{cc|c} 1 & x_2/x_1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and we find $\vec{v}_1 = [-x_2\, x_1]^T$. Similarly, for $\lambda = x_1^2 + x_2^2$, we find

$$\left[\begin{array}{cc|c} 1 & -x_1/x_2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and $\vec{v}_2 = [x_1\, x_2]^T$

Exercise 3.6:

In this transition region, we have $\psi(r) = g(r)$. From this, it follows that

$$\gamma_m = \frac{rg'(r)}{g(r)} \text{ and } \gamma_M = \frac{g(r)}{rg'(r)}$$

As $r \to \frac{1}{2} + \delta$, $g(r) \to r$ and $g'(r) \to 1$. As this transition occurs, $\gamma_m \to 1$ and $\gamma_M \to 1$. In general, the construction of $\psi(r)$ as a twice-continuously differentiable function guarantees a smooth transition from region to region ($\rho < r \leq \frac{1}{2}$ to $\frac{1}{2} < r < \frac{1}{2} + \delta$ to $\frac{1}{2} + \delta \leq r \leq 1$).

Exercise 3.7:

In three dimensions, we have

$$D\phi(\vec{x}) = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \partial y_1/\partial x_3 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \partial y_3/\partial x_1 & \partial y_3/\partial x_2 & \partial y_3/\partial x_3 \end{bmatrix}$$

Lemma 3.1 still holds and similar to Exercise 3.2, we may verify that $\sigma(\vec{y})$ is positive definite. For an arbitrary vector $\vec{w} = [w_1\, w_2\, w_3]^T$ in $\mathbb{R}^3$,

$$\begin{aligned} \vec{w}^T\sigma(\vec{y})\vec{w} = &\Big(w_1\frac{\partial y_1}{\partial x_1} + w_2\frac{\partial y_2}{\partial x_1} + w_3\frac{\partial y_3}{\partial x_1}\Big)^2 \\ &+\Big(w_1\frac{\partial y_1}{\partial x_2} + w_2\frac{\partial y_2}{\partial x_2} + w_3\frac{\partial y_3}{\partial x_2}\Big)^2 \\ &+\Big(w_1\frac{\partial y_1}{\partial x_3} + w_2\frac{\partial y_2}{\partial x_3} + w_3\frac{\partial y_3}{\partial x_3}\Big)^2 \end{aligned}$$

where the presence of the squares guarantees that the expression is not negative. To verify that the expression is not zero, note that the terms in parentheses must equal zero simultaneously. This results in a homogeneous system of equations similar to the one in Exercise 3.2. The coefficient matrix is $D\phi(\vec{x})$ and since $\det(D\phi(\vec{x})) \neq 0$, we only have the trivial solution $w_1 = w_2 = w_3 = 0$. Thus, $\sigma(\vec{y})$ is positive definite. For $\phi(\vec{x}) = \dfrac{\psi(||\vec{x}||}{||\vec{x}||}\vec{x}$, we have

$$D\phi = (\psi'(r)/r^2 - \psi(r)/r^3)\begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{bmatrix} + (\psi(r)/r)I$$

The work from Exercise 3.4 still holds, but we must find the eigenvalues and eigenvectors of this matrix as we did in Exercise 3.5. We find that for $D\phi$, we have eigenvalues $\mu_1 = \psi(r)/r$ with multiplicity 2 and $\mu_2 = (\psi'(r) - \psi(r)/r)) + \psi(r)/r$ with eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We also find that, as in two dimensions, $\det(D\phi) = \mu_1\mu_2$, $\gamma_m = \frac{\mu_1^2}{\mu_1\mu_2}$, and $\gamma_M = \frac{\mu_2^2}{\mu_1\mu_2}$
Exercise 3.8 is covered in the next section.

2 Imaging Conductors

Exercise 3.8 introduces the concept of cloaking a region of homogeneous isotropic conductivity α (not a void). We note that in this situation, for $r \leq \rho$, we have $\alpha\Delta u_0 = 0$ where u_0 is defined for $r = 0$. In the region $\rho \leq r \leq 1$, we have $\gamma\Delta u = 0$ and since we would like to cloak the region of isotropic conductivity α as a region of isotropic conductivity 1, we have $\Delta u = 0$. The conditions we use to find the coefficients for each of these solutions are as follows:

1. $u = f$ on $\partial\Omega$ (Dirichlet Boundary Condition)
2. $u_0(\rho, \theta) = u(\rho, \theta)$ (smooth transition from u_0 to u)
3. $\frac{\partial u}{\partial \mathbf{n}}(\rho, \theta) = \alpha\frac{\partial u_0}{\partial \mathbf{n}}(\rho, \theta)$ (flux in = flux out; note that $\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial r}$ for the normal derivative.

Note that conditions 2 and 3 replace the Neumann data condition on ∂D that we had for imaging a void. $u(r, \theta)$ has the same form as it had for imaging voids:

$$u(r, \theta) = c_0 + d_0 \ln r + \sum_{k \in \mathbb{Z}/\{0\}} (c_k r^{|k|} + d_k r^{-|k|})e^{ik\theta}$$

From the Dirichlet Boundary Condition $u(1, \theta) = f(\theta)$, we have

$$c_0 + \sum_{k \in \mathbb{Z}/\{0\}} (c_k + d_k)e^{ik\theta} = f(\theta)$$

As we did for a void, we assume f is continuous and piecewise differentiable so that the Fourier series converges pointwise to f. Expanding f as a Fourier Series

$$f(\theta) = \sum_{k \in \mathbb{Z}} f_k e^{ik\theta} \quad f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{ik\theta}$$

We find our first expression for the coefficients of u:

$$c_0 = f_0 \quad c_k + d_k = f_k \text{ for } k \in \mathbb{Z}/\{0\} \tag{1}$$

Since u_0 is defined at $r = 0$, the solution for $\alpha\Delta u_0 = 0$ is simply found by eliminating the terms that have singularities at $r = 0$:

$$u_0(r, \theta) = a_0 + \sum_{k\in\mathbb{Z}/\{0\}} a_k r^{|k|} e^{ik\theta}$$

Using the smoothness condition $u_0(\rho, \theta) = u(\rho, \theta)$, we find

$$a_0 = c_0 + d_0 \ln(\rho) \tag{2}$$

$$c_k\rho^{|k|} + d_k\rho^{-|k|} = a_k\rho^{|k|} \text{ for } k \in \mathbb{Z}/\{0\} \tag{3}$$

Using the flux condition $\frac{\partial u}{\partial r}(\rho, \theta) = \alpha\frac{\partial u_0}{\partial r}(\rho, \theta)$, we find

$$d_0 = 0 \tag{4}$$

$$\alpha a_k\rho^{|k|-1} = c_k\rho^{|k|-1} - d_k\rho^{-|k|-1} \tag{5}$$

Using (1), (2), and (4), we find $a_0 = c_0 = f_0$ and $d_0 = 0$. We may employ (1), (3), and (5) to obtain the coefficients a_k, c_k, and d_k. We find

$$a_k = \frac{2f_k}{1 + \alpha + (1-\alpha)\rho^{2|k|}} \text{ for } \mathbb{Z}/\{0\} \tag{6}$$

$$c_k = \frac{(1+\alpha)f_k}{1 + \alpha + (1-\alpha)\rho^{2|k|}} \text{ for } \mathbb{Z}/\{0\} \tag{7}$$

$$d_k = \frac{(1-\alpha)f_k\rho^{2|k|}}{1 + \alpha + (1-\alpha)\rho^{2|k|}} \text{ for } \mathbb{Z}/\{0\} \tag{8}$$

Thus, we have

$$u_0(r, \theta) = f_0 + \sum_{k\in\mathbb{Z}/\{0\}} \frac{2f_k r^{|k|} e^{ik\theta}}{1 + \alpha + (1-\alpha)\rho^{2|k|}} \tag{9}$$

$$u(r, \theta) = f_0 + \sum_{k\in\mathbb{Z}/\{0\}} \Big(\frac{(1+\alpha)f_k r^{|k|}}{1 + \alpha + (1-\alpha)\rho^{2|k|}} + \frac{(1-\alpha)f_k\rho^{2|k|}r^{-|k|}}{1 + \alpha + (1-\alpha)\rho^{2|k|}}\Big)e^{ik\theta} \tag{10}$$

As we did in the Bad Cloaking section, we consider the error in this cloak. Let u_1 be the solution to the Laplacian on Ω with Dirichlet data $u_1 = f$ on $\partial\Omega$ (when Ω is empty). We compute the difference between the Neumann data for u and u_1 (note that u_1 resembles u_0):

$$\frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1, \theta) = \sum_{k\in\mathbb{Z}} \frac{2(\alpha - 1)|k|\rho^{2|k|}f_k}{1 + \alpha + (1-\alpha)\rho^{2|k|}} e^{ik\theta}$$

using the fact that $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial r}$ on the boundary $r = 1$. Now, computing the $L^2(\partial\Omega)$ norm as

$$\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1, \theta)\Big|\Big|^2_{L^2(\partial\Omega)} = \int_0^{2\pi} \Big|\frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1, \theta)\Big|^2 d\theta$$

Using Parseval's Identity, we find

$$\begin{aligned}\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1, \theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1, \theta)\Big|\Big|^2_{L^2(\partial\Omega)} &= \frac{4k^2\rho^{4|k|}}{(1 + \alpha + (1-\alpha)\rho^{2|k|})^2}|f_k|^2 \\ &\leq 4k^2\rho^{4|k|}|f_k|^2 \\ &\leq 4\rho^4\Big|\Big|\frac{\partial u_1}{\partial \mathbf{n}}(1, \theta)\Big|\Big|^2_{L^2(\partial\Omega)}\end{aligned}$$

The second line follows from the fact that for $0 < \rho < 1$, $(1 + \alpha + (1 - \alpha)\rho^{2|k|})^2 > 1$.
The third line follows from the $L^2(\partial\Omega)$ norm of $u_1(r, \theta)$ and that for $0 < \rho < 1$ and $k \geq 1$, $\rho^{4|k|} \leq \rho^4$
Finally, taking the square root, we find that the error in the cloaking is proportional to ρ^2 as it was for imaging a void.

$$\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1,\theta)\Big|\Big|_{L^2(\partial\Omega)} \leq 2\rho^2 \Big|\Big|\frac{\partial u_1}{\partial \mathbf{n}}(1,\theta)\Big|\Big|_{L^2(\partial\Omega)} \tag{11}$$

3 Cloaking Conductors

Here we adapt the process from cloaking a void to cloak the region of isotropic conductivity α. Let Ω_ρ be the open annulus $\Omega/\overline{B_\rho(0)}$ with $\rho \in (0, 1/3)$ and let u be a twice continuously differentiable solution to the Laplacian on Ω_ρ with $u = f$ on $\partial\Omega_\rho$ and $\frac{\partial u}{\partial r}(\rho, \theta) = \alpha \frac{\partial u_0}{\partial r}(\rho, \theta)$ and $u_0(\rho, \theta) = u(\rho, \theta)$ where u_0 is the solution to the Laplacian on $B_\rho(0)$
We consider the invertible mapping $\phi : \Omega_\rho \to \Omega_{1/3}$ where ϕ and ϕ' are twice continuously differentiable. ϕ will map points $\vec{x} = (x_1, x_2)$ in Ω_ρ to $\vec{y} = (y_1, y_2)$ in $\Omega_{1/3}$ such that for $||\vec{x}|| = \rho$, we have $||\vec{y}|| = 1/3$ and for $||\vec{x}|| = 1$, $||\vec{y}|| = 1$. Lemma 3.1 still holds, that is, for a twice continuously differentiable function v on $\Omega_{1/3}$ (where $v(\vec{y}) = u(\phi^{-1}(\vec{y}))$,

$$\nabla \cdot \sigma(\vec{y})\nabla v = 0$$

in $\Omega_{1/3}$ where $\sigma(\vec{y}) = [D(\phi(\vec{x})(D\phi(\vec{x})^T]/|\det{(D\phi(\vec{x})|}$ at $\vec{x} = \phi^{-1}(\vec{y})$ where the nonsingular matrix $D\phi(\vec{x})$ is defined as

$$D\phi(\vec{x}) = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{bmatrix}$$

We must construct the cloak such that the problem on Ω_ρ is conserved. We have

$$\phi(\vec{x}) = \frac{\psi(||\vec{x}||}{||\vec{x}||}\vec{x}$$

so that $\vec{y} = \phi(\vec{x})$ means $y_1 = \frac{\psi(||\vec{x}||}{||\vec{x}||}x_1$ and $y_2 = \frac{\psi(||\vec{x}||}{||\vec{x}||}x_2$.
The conditions we must consider are as follows:

1. $\psi(\rho) = 1/3$
2. For some $\delta \in (0, 2/3)$, $\psi(r) = r$ for $\frac{1}{3} + \delta \leq r \leq 1$
3. ψ is twice continuously differentiable with $\psi'(r) \geq d_0 > 0$ for some d_0 so that ψ will be strictly increasing and invertible.
4. Unlike the problem with the void, we must define ψ on $0 \leq r \leq \rho$

With these conditions, we may construct $\psi(r)$ as follows:

$$\psi(r) = \begin{cases} \frac{1}{3} + (r - \rho) & 0 < r \leq \rho + \epsilon \\ g(r) & \rho + \epsilon < r < \frac{1}{3} + \delta \\ r & \frac{1}{3} + \delta < r < 1 \end{cases}$$

where $\epsilon \in (0, 1/3)$ guarantees that $\psi(r)$ smoothly transitions from $r < \rho$ to $r > \rho$ through $V_\epsilon(\rho)$. Note that the Laplacian is satisfied for $r < \rho$. Additionally, we have $\lim_{\epsilon\to 0^-} \psi(\rho + \epsilon) = \lim_{\epsilon\to 0^+} \psi(\rho + \epsilon)$ $\lim_{\epsilon\to 0^-} \psi'(\rho + \epsilon) = \lim_{\epsilon\to 0^+} \psi'(\rho + \epsilon)$ so the smoothness conditions: $\frac{\partial u}{\partial r}(\rho, \theta) = \alpha \frac{\partial u_0}{\partial r}(\rho, \theta)$ and $u(\rho, \theta) = u_0(\rho, \theta)$ are conserved in this change of variables. As we had in Exercise 3.3, there are several conditions for $g(r)$:

1. $g(\rho + \epsilon) = \frac{1}{3} + \epsilon$
2. $g'(\rho + \epsilon) = 1$
3. $g''(\rho + \epsilon) = 0$

4. $g(\frac{1}{3}+\delta) = \frac{1}{3}+\delta$
5. $g'(\frac{1}{3}+\delta) = 1$
6. $g''(\frac{1}{3}+\delta) = 0$

As proposed in the paper, we may construct $g(r)$ as a fifth degree polynomial of the form $a_5r^5 + a_4r^4 + a_3r^3 + a_2r^2 + a_1r + a_0$ and a computer algebra system may be employed with particular values for ϵ and δ.
Behavior in the Cloaking Region
We may also look at the behavior in the inner cloaking region $\frac{1}{3} \le ||\vec{y}|| \le \frac{1}{3}+\epsilon$ (corresponding to $\rho \le ||\vec{x}|| \le \rho+\epsilon$). We still have the same expression for $D\phi$:

$$D\phi = (\psi'(r)/r^2 + \psi(r)/r^3)\begin{bmatrix} x_1^2 & x_1x_2 \\ x_1x_2 & x_2^2 \end{bmatrix} + (\psi(r)/r)I$$

Using $\sigma = (D\phi)^2/|\det D\phi|$ in conjunction with Exercises 3.4 and 3.5, we find the eigenvectors $\vec{v}_1 = [x_1, x_2]^T$ and $\vec{v}_2 = [-x_2, x_1]^T$, with eigenvalues $\mu_1 = \psi'(r)$ and $\mu_2 = \psi(r)/r$. $\det D\phi = \mu_1\mu_2$ so for the eigenvalues of σ, we find

$$\gamma_m = \frac{r\psi'(r)}{\psi(r)} \quad \gamma_M = \frac{\psi(r)}{r\psi'(r)}$$

Using the constructed $\psi(r)$:

$$\gamma_m = \frac{3r}{1+3r-3\rho} \quad \gamma_M = \frac{1+3r-3\rho}{3r}$$

Our conclusions regarding these eigenvalues are very similar to those made in the paper. At $r=\rho$, we find

$$\gamma_m = 3\rho \quad \gamma_M = \frac{1}{3\rho}$$

where as $\rho \to 0^+$, $\gamma_m \to 0^+$ and γ_M becomes arbitrarily large. As you can see, despite the difference in the region being one of isotropic conductivity rather than a void, the behavior of the cloak is similar.

4 Conclusion

The paper by Kurt Bryan and Tanya Leise provides a good base understanding of the cloaking problem. The paper covers a simplified example of a PDE for which an explicit solution can be obtained using separation of variables. However, it occurs more often than not that these types of explicit solutions are not available. Nonetheless, the solutions discussed make it easy to study the behavior of the solution in certain regions such as the inner cloaking region. Using the two problems discussed here, cloaking a void and cloaking a region of isotropic conductivity, one could rig up a problem with n concentric circles and certain boundary conditions to cloak some regions as voids and others as regions as ones with a certain isotropic conductivity. The problems could be tedious but much of the work will be the same. Applying the Dirichlet data condition $u = f$ on $\partial\Omega$ and either the Neumann data condition $\frac{\partial u}{\partial \mathbf{n}} = 0$ or a flux condition $\alpha_k \frac{\partial u_k}{\partial \mathbf{n}} = \alpha_{k-1}\frac{\partial u_{k-1}}{\partial \mathbf{n}}$ (where α_k and α_{k-1} denotes the isotropic conductivity in regions corresponding to solutions u_k and u_{k-1}) and solving the Laplacian $\Delta u = 0$ will yield certain functions in each notable region.