Cloaking Research

Rohit Narayanan

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1 Exercises

An Enumeration of the Exercises in Kurt Bryan and Tanya Leise's paper *Impedance Imaging, Inverse Problems, and Harry Potter's Cloak*

Exercise 2.1:

First we must verify that the function $u(x_1, x_2)$ satisfies the Laplacian $\Delta u = 0$ coupled with the Dirichlet data $u(x_1, x_2) = f(\theta)$ on $\partial\Omega$. Since $\dfrac{\partial u}{\partial x_1} = a$ and $\dfrac{\partial u}{\partial x_2} = b$, we have $\Delta u = \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0$. In verifying the Dirichlet data, we find $ax_1 + bx_2 + c = a\cos\theta + b\sin\theta + c$ and it is clear that the parametrization $x_1 = \cos\theta$ and $x_2 = \sin\theta$ yields the desired $u(x_1, x_2) = f(\theta)$

Exercise 2.2:

An isotropic conductor has no directional properties. We seek to find an orthogonal basis of vectors that span $\mathbb{R}^2$ of magnitude γ. The simplest way of doing this is simply multiplying γ to the two unit vectors $\vec{e_1}$ and $\vec{e_2}$, yielding

$$\vec{v_1} = \begin{bmatrix} \gamma \\ 0 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ \gamma \end{bmatrix}$$

To check that $\vec{v_1}$ and $\vec{v_2}$ satisfy the conditions, we verify the following:

1. $\vec{v_1} \times \vec{v_2} = \gamma \cdot 0 + 0 \cdot \gamma = 0$
2. $\sqrt{\vec{v_1} \cdot \vec{v_1}} = \sqrt{\gamma^2 + 0} = \gamma$
3. $\sqrt{\vec{v_2} \cdot \vec{v_2}} = \sqrt{0 + \gamma^2} = \gamma$
4. $c_1\vec{v_1} + c_2\vec{v_2} = 0$ only has the trivial solution $c_1 = c_2 = 0$ therefore $\text{span}([\vec{v_1}, \vec{v_2}]) = \mathbb{R}^2$

Finally, we write $\sigma = [\vec{v_1}|\vec{v_2}] = \gamma I$:

$$\sigma = \begin{bmatrix} \gamma & 0 \\ 0 & \gamma \end{bmatrix}$$

Exercise 2.3:

We use $\vec{v_M}$ and $\vec{v_m}$ as the two orthonormal eigenvectors corresponding to the eigenvalues γ_M and γ_m in the decomposition of σ. Then, using $\sigma = PDP^{-1} = PDP^T$, where P is the matrix with the eigenvectors and D is the diagonal matrix with the eigenvalues of σ (in the same order as the eigenvectors in P), we have

$$\sigma = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \gamma_M & 0 \\ 0 & -\gamma_m \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

Simplifying this, we obtain

$$\sigma = \frac{1}{2} \begin{bmatrix} \gamma_M + \gamma_m & \gamma_M - \gamma_m \\ \gamma_M - \gamma_m & \gamma_M + \gamma_m \end{bmatrix}$$

As $\gamma_m \to 0^+$,

$$\sigma \to \frac{1}{2} \begin{bmatrix} \gamma_M & \gamma_M \\ \gamma_M & \gamma_M \end{bmatrix}$$

Exercise 2.4:
First, we calculate $||\vec{v}||^2$. $||\vec{v}||^2 = \vec{v} \cdot \vec{v}$. We use $\vec{v} = \sum_{k=1}^{n} \alpha_k \vec{v}_k$ where $\vec{v}_k$ are orthonormal eigenvectors for σ and α_k are some scalars. Then, we may compute $||\vec{v}||^2 = (\sum_{i=1}^{n} \alpha_i \vec{v}_i)(\sum_{k=1}^{n} \alpha_k \vec{v}_k)$. Now, noting that the vectors $\vec{v}_k$ are orthonormal, $\vec{v}_i \cdot \vec{v}_k = 0$ if $i \neq k$ and $\vec{v}_i \cdot \vec{v}_k = 1$ if $i = k$. Thus, we have $||\vec{v}||^2 = \sum_{k=1}^{n} \alpha_k^2$ and since we have fixed $||\vec{v}|| = 1$, $\sum_{k=1}^{n} \alpha_k^2 = 1$.
Now we consider $||\sigma\vec{v}||^2$ for an arbitrary vector $\vec{v}$. $||\sigma\vec{v}||^2 = \sigma\vec{v} \cdot \sigma\vec{v}$. $\sigma\vec{v} \cdot \sigma\vec{v} = (\sum_{i=1}^{n} \alpha_i \sigma\vec{v}_i)(\sum_{k=1}^{n} \alpha_k \sigma\vec{v}_k)$. Note that the $\vec{v}_i$ and $\vec{v}_k$ are the eigenvectors of σ so we have $\sigma\vec{v}_k = \lambda_k \vec{v}_k$. Using this and the orthonormal nature of the vectors, we have $||\sigma\vec{v}||^2 = \sum_{k=1}^{n} \alpha_k^2 \lambda_k^2$.
Finally, we must show that the previous expression is less than or equal to $||\sigma\vec{v}_n||^2$, where $\vec{v}_n$ is the eigenvector corresponding to the largest eigenvalue λ_n. Since $\sigma\vec{v}_n = \lambda_n \vec{v}_n$ and $||\vec{v}_n|| = 1$, we have $||\sigma\vec{v}_n||^2 = ||\lambda_n \vec{v}_n||^2 = \lambda_n^2$. Note that since λ_n is the largest eigenvalue, we have the inequality

$$\sum_{k=1}^{n} \alpha_k^2 \lambda_k^2 \leq \sum_{k=1}^{n} \alpha_k^2 \lambda_n^2 = \lambda_n^2$$

where the last equality follows from fixing $||\vec{v}|| = \sum_{k=1}^{n} \alpha_k^2 = 1$. Thus, we have

$$||\sigma\vec{v}||^2 \leq ||\sigma\vec{v}_n||^2$$

Example 2.5 is explained in the paper so it is omitted here.

Exercise 2.6:
We must verify three conditions: the Laplacian, the Neumann data on ∂D, and the Dirichlet data on $\partial\Omega$. It is easier to verify the Laplacian in polar coordinates, using $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$. Then, we have

$$u(r,\theta) = (\sin\theta + \cos\theta)\Big(\frac{4r^2+1}{5r}\Big)$$

In polar coordinates, the Laplacian becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Computing the relevant derivatives, we have

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = (-\sin\theta - \cos\theta)\Big(\frac{4r^2+1}{5r^3}\Big)$$

$$\frac{1}{r}\frac{\partial u}{\partial r} = (\sin\theta + \cos\theta)\Big(\frac{4r^2-1}{5r^3}\Big)$$

$$\frac{\partial^2 u}{\partial r^2} = (\sin\theta + \cos\theta)\Big(\frac{2}{5r^3}\Big)$$

Adding these terms yields the expected verification of the Laplacian.
On ∂D, we must have $\frac{\partial u}{\partial \mathbf{n}} = 0$. We use $\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial r}$. Setting $\frac{\partial u}{\partial r} = 0$, we find $r = \frac{1}{2}$ which we expect since we have removed a ball $D = B_{1/2}(0)$ (a ball of radius 1/2 centered at 0).
Finally, we check the Dirichlet data on $\partial\Omega$. To do so, we simply show that $u(1,\theta) = f(\theta) = \cos\theta + \sin\theta$. Plugging into $u(r,\theta)$, we find $u(1,\theta) = \cos\theta + \sin\theta$

Exercise 2.7:
The paper covers the general solution to the Laplacian on an annulus:

$$u(r,\theta) = \sum_{k\in\mathbb{Z}} \Big(\frac{f_k}{1+\rho^{2|k|}} r^{|k|} e^{ik\theta} + \frac{\rho^{2|k|} f_k}{1+\rho^{2|k|}} r^{-|k|} e^{ik\theta}\Big)$$

This solution already accounts for $\frac{\partial u}{\partial \mathbf{n}}(\rho,\theta) = 0$ so the only task left is to obtain f_k for $f(\theta) = \cos\theta$. For f_k, we have the expression

$$f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-ik\theta}\,d\theta$$

To evaluate this integral, we may use $e^{-ik\theta} = \cos k\theta - i\sin k\theta$. Then, we have two separate integrals in the expression for f_k:

$$\int_0^{2\pi} \cos\theta\cos k\theta\,d\theta$$

and

$$\int_0^{2\pi} \cos\theta\sin k\theta\,d\theta$$

The set $\{1, \sin\theta, \cos\theta, \sin 2\theta, \cos 2\theta\cdots\}$ is mutually orthogonal so the only nonzero integral occurs when $k = 1$:

$$f_1 = \frac{1}{2\pi}\int_0^{2\pi}\cos^2\theta\,d\theta = \frac{1}{2}$$

Plugging this into the expression for $u(r,\theta)$, we find

$$u(r,\theta) = \frac{e^{i\theta}}{2(1+\rho^2)}\Big(r + \frac{\rho^2}{r}\Big)$$

Exercise 2.8:

We must simply utilize the work from Exercise 2.7 showing that for the Dirichlet boundary condition $u(1,\theta) = f(\theta) = \cos\theta$, the only Fourier coefficient that is nonzero is $f_1 = \frac{1}{2}$. Using this in Equation 2.18 with $k = 1$, we have

$$\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta)\Big|\Big|^2_{L^2\Omega} = \frac{\rho^4}{4(1+\rho^2)^2}$$

And taking the square root of both sides, we obtain

$$\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta)\Big|\Big|_{L^2\Omega} = \frac{\rho^2}{2(1+\rho^2)}$$

Exercise 2.9:

If $f_1 \neq 0$, then we have

$$\frac{\partial u}{\partial \mathbf{n}}(1,\theta) = \frac{1-\rho^2}{1+\rho^2}f_1e^{i\theta} + \sum_{k\in\mathbb{Z}/\{1\}}\frac{|k|(1-\rho^{2|k|})}{1+\rho^{2|k|}}f_ke^{ik\theta}$$

However, note that upon evaluating the integral expression for I, we observe that the orthogonality of the functions $e^{ik\theta}$ guarantees that

$$\int_0^{2\pi} e^{ik_1\theta}e^{-ik_2\theta}\,d\theta = \begin{cases} 0 & k_1 \neq k_2 \\ 1 & k_1 = k_2 \end{cases}$$

And thus, the only nonzero term for the integral occurs when $k = 1$:

$$I = \int_0^{2\pi}\frac{\partial u}{\partial \mathbf{n}}(1,\theta)e^{-i\theta}\,d\theta = \int_0^{2\pi}\frac{1-\rho^2}{1+\rho^2}f_1\,d\theta$$

Evaluating this expression, we find the desired relationship

$$\frac{1-\rho^2}{1+\rho^2} = \frac{I}{2\pi f_1}$$

Exercise 2.10:

To obtain the inequality, we omit all terms except for $k = -1, 1$. Using $k = -1, 1$ in Equation 2.16, we have

$$\left|\left|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta)\right|\right|^2_{L^2\Omega} \geq \frac{\rho^4}{(1+\rho^2)^2}|f_{-1}|^2 + \frac{\rho^4}{(1+\rho^2)^2}|f_1|^2$$

If f is real, then $f_{-1} = \overline{f_1}$ so $|f_{-1}|^2 = |f_1|^2$ and:

$$\left|\left|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta)\right|\right|^2_{L^2\Omega} \geq \frac{2\rho^4}{(1+\rho^2)^2}|f_1|^2$$

Taking the square root and noting that for $\rho \leq 1$, $1+\rho^2 \leq 2$, we find the final expression

$$\left|\left|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_0}{\partial \mathbf{n}}(1,\theta)\right|\right|_{L^2(\Omega)} \geq \frac{\sqrt{2}}{2}|f_1|\rho^2$$

The order of the error in the difference in the Neumann data remains proportional to ρ^2 and so it is optimal.

Lemma 3.1 is stated and proved in the paper so it is omitted here.

Exercise 3.2:
We have defined $\sigma(\vec{y})$ as

$$\sigma(\vec{y}) = \frac{D\phi(\vec{x})(D\phi(\vec{x}))^T}{det(D\phi(\vec{x}))}$$

Simple computation yields

$$\sigma(\vec{y}) = \frac{\begin{bmatrix} \left(\frac{\partial y_1}{\partial x_1}\right)^2 + \left(\frac{\partial y_1}{\partial x_2}\right)^2 & \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_1} + \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_2} \\ \frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_1} + \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_2} & \left(\frac{\partial y_2}{\partial x_1}\right)^2 + \left(\frac{\partial y_2}{\partial x_2}\right)^2 \end{bmatrix}}{\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1}}$$

Then, we consider an arbitrary vector $\vec{w} = [w_1\, w_2]$. We omit $\det \sigma(\vec{y})$ since the matrix $\sigma(\vec{y})$ is nonsingular and since the partial derivatives are continuous, then the Jacobian is either always positive or always negative. Computing $\vec{w}^T\sigma(\vec{y})\vec{w}$ yields

$$\vec{w}^T\sigma(\vec{y})\vec{w} = \left(w_1\frac{\partial y_1}{\partial x_1} + w_2\frac{\partial y_2}{\partial x_1}\right)^2 + \left(w_1\frac{\partial y_1}{\partial x_2} + w_2\frac{\partial y_2}{\partial x_2}\right)^2$$

where the presence of squares (and lack of imaginary terms!) guarantees $\vec{w}^T\sigma(\vec{y})\vec{w}$ is not negative. We must confirm that $\vec{w}^T\sigma(\vec{y})\vec{w} \neq 0$. We do so using a proof by contradiction. If $\vec{w}^T\sigma(\vec{y})\vec{w} = 0$, then we must have

$$w_1\frac{\partial y_1}{\partial x_1} + w_2\frac{\partial y_2}{\partial x_1} = 0$$

and

$$w_1\frac{\partial y_1}{\partial x_2} + w_2\frac{\partial y_2}{\partial x_2} = 0$$

simultaneously. However, note that the determinant of the coefficient matrix is $\det D\phi(\vec{x})$. Thus, the fact that $D\phi(\vec{x})$ is nonsingular ($\det D\phi(\vec{x}) \neq 0$) guarantees that the only solution to this system is the trivial one $w_1 = w_2 = 0$. Therefore, $\sigma(\vec{y})$ is positive definite.

Exercise 3.3:
We must construct $g(r)$ such that $\psi(r)$ is twice-continuously differentiable.

$$\psi(r) = \begin{cases} \frac{1}{2} + \frac{\delta}{1-2\rho}(r-\rho) & \rho \leq r < \frac{1}{2} \\ g(r) & \frac{1}{2} < r < \frac{1}{2}+\delta \\ r & \frac{1}{2}+\delta \leq r \leq 1 \end{cases}$$

We have several conditions that $g(r)$ must satisfy:

1. $g(\frac{1}{2}) = \frac{1}{2} + \frac{\delta}{1-2\rho}(\frac{1}{2} - \rho)$
2. $g'(\frac{1}{2}) = \frac{1}{2} + \frac{\delta}{1-2\rho}$
3. $g(\frac{1}{2} + \delta) = \frac{1}{2} + \delta$
4. $g'(\frac{1}{2} + \delta) = 1$
5. $g''(\frac{1}{2}) = 0$
6. $g''(\frac{1}{2} + \delta) = 0$

We may try a fifth degree polynomial of the form $g(r) = a_5r^5 + a_4r^4 + a_3r^3 + a_2r^2 + a_1r + a_0$ where $g'(r) = 5a_5r^4 + 4a_4r^3 + 3a_3r^2 + 2a_2r + a_1$ and $g''(r) = 20a_5r^3 + 12a_4r^2 + 6a_3r + 2a_2$. We use the previously enumerated conditions to find expressions for the constants $a_0 \cdots a_5$ (a computer algebra program may be employed here).

Exercise 3.4:
Since $\vec{v}$ is an eigenvector corresponding to the eigenvalue μ for the matrix A, we have $A\vec{v} = \mu\vec{v}$. Thus, if we define $B = A^2/|det(A)|$, we have:

$$B\vec{v} = \frac{A}{|det(A)|}A\vec{v} = \mu\frac{A\vec{v}}{|det(A)|} = \frac{\mu^2}{|det(A)|}\vec{v}$$

And $\vec{v}$ is an eigenvector for B corresponding to the eigenvalue $\lambda = \mu^2/|det(A)|$

Exercise 3.5:
Here we simply follow the steps in finding eigenvalues and eigenvectors. First, we obtain the eigenvalues:

$$A - \lambda I = \begin{bmatrix} x_1^2 - \lambda & x_1x_2 \\ x_1x_2 & x_2^2 - \lambda \end{bmatrix}$$

Then, using the characteristic equation, we find $\det(A - \lambda I) = \lambda^2 - (x_1^2 + x_2^2)\lambda = 0$ and $\lambda = 0, x_1^2 + x_2^2$. Next, we find the null space using each of these eigenvalues.

$$A - 0I = \left[\begin{array}{cc|c} x_1^2 - \lambda & x_1x_2 & 0 \\ x_1x_2 & x_2^2 - \lambda & 0 \end{array}\right]$$

Row reduction yields

$$\left[\begin{array}{cc|c} 1 & x_2/x_1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and we find $\vec{v}_1 = [-x_2\, x_1]^T$. Similarly, for $\lambda = x_1^2 + x_2^2$, we find

$$\left[\begin{array}{cc|c} 1 & -x_1/x_2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

and $\vec{v}_2 = [x_1\, x_2]^T$

Exercise 3.6:
In this transition region, we have $\psi(r) = g(r)$. From this, it follows that the smallest eigenvalue γ_m and the largest eigenvalue γ_M are:

$$\gamma_m = \frac{rg'(r)}{g(r)} \text{ and } \gamma_M = \frac{g(r)}{rg'(r)}$$

As $r \to \frac{1}{2} + \delta$, $g(r) \to r$ and $g'(r) \to 1$. As this transition occurs, $\gamma_m \to 1$ and $\gamma_M \to 1$. In general, the construction of $\psi(r)$ as a twice-continuously differentiable function guarantees a smooth transition from region to region ($\rho < r \leq \frac{1}{2}$ to $\frac{1}{2} < r < \frac{1}{2} + \delta$ to $\frac{1}{2} + \delta \leq r \leq 1$).

Exercise 3.7:

In three dimensions, we have

$$D\phi(\vec{x}) = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 & \partial y_1/\partial x_3 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 & \partial y_2/\partial x_3 \\ \partial y_3/\partial x_1 & \partial y_3/\partial x_2 & \partial y_3/\partial x_3 \end{bmatrix}$$

Lemma 3.1 still holds and similar to Exercise 3.2, we may verify that $\sigma(\vec{y})$ is positive definite. For an arbitrary vector $\vec{w} = [w_1\, w_2\, w_3]^T$ in $\mathbb{R}^3$,

$$\begin{aligned} \vec{w}^T\sigma(\vec{y})\vec{w} = &\Big(w_1\frac{\partial y_1}{\partial x_1} + w_2\frac{\partial y_2}{\partial x_1} + w_3\frac{\partial y_3}{\partial x_1}\Big)^2 \\ &+\Big(w_1\frac{\partial y_1}{\partial x_2} + w_2\frac{\partial y_2}{\partial x_2} + w_3\frac{\partial y_3}{\partial x_2}\Big)^2 \\ &+\Big(w_1\frac{\partial y_1}{\partial x_3} + w_2\frac{\partial y_2}{\partial x_3} + w_3\frac{\partial y_3}{\partial x_3}\Big)^2 \end{aligned}$$

where the presence of the squares guarantees that the expression is not negative. To verify that the expression is not zero, note that the terms in parentheses must equal zero simultaneously. This results in a homogeneous system of equations similar to the one in Exercise 3.2. The coefficient matrix is $D\phi(\vec{x})$ and since $\det(D\phi(\vec{x})) \neq 0$, we only have the trivial solution $w_1 = w_2 = w_3 = 0$. Thus, $\sigma(\vec{y})$ is positive definite. For $\phi(\vec{x}) = \dfrac{\psi(||\vec{x}||}{||\vec{x}||}\vec{x}$, we have

$$\begin{aligned} D\phi = (\psi'(r)/r^2 - \psi(r)/r^3) &\begin{bmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_1x_2 & x_2^2 & x_2x_3 \\ x_1x_3 & x_2x_3 & x_3^2 \end{bmatrix} \\ &+(\psi(r)/r)I \end{aligned}$$

The work from Exercise 3.4 still holds, but we must find the eigenvalues and eigenvectors of this matrix as we did in Exercise 3.5. We find that for $D\phi$, we have eigenvalues $\mu_1 = \psi(r)/r$ with multiplicity 2 and $\mu_2 = (\psi'(r) - \psi(r)/r)) + \psi(r)/r$ with eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} \; \vec{v}_2 = \begin{bmatrix} -x_3 \\ 0 \\ x_1 \end{bmatrix} \; \vec{v}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We also find that, as in two dimensions, $\det(D\phi) = \mu_1\mu_2$, $\gamma_m = \frac{\mu_1^2}{\mu_1\mu_2}$, and $\gamma_M = \frac{\mu_2^2}{\mu_1\mu_2}$

Exercise 3.8 is covered in the next section.

2 Imaging Conductors

Exercise 3.8 introduces the concept of cloaking a region of homogeneous isotropic conductivity α (not a void). We note that in this situation, for $r \leq \rho$, we have $\alpha\Delta u_0 = 0$ where u_0 is defined for $r = 0$. In the region $\rho \leq r \leq 1$, we have $\gamma\Delta u = 0$ and since we would like to cloak the region of isotropic conductivity α as a region of isotropic conductivity 1, we have $\Delta u = 0$. The conditions we use to find the coefficients for each of these solutions are as follows:

1. $u = f$ on $\partial\Omega$ (Dirichlet Boundary Condition)
2. $u_0(\rho, \theta) = u(\rho, \theta)$ (smooth transition from u_0 to u)
3. $\frac{\partial u}{\partial \mathbf{n}}(\rho, \theta) = \alpha\frac{\partial u_0}{\partial \mathbf{n}}(\rho, \theta)$ (flux in = flux out; note that $\frac{\partial}{\partial \mathbf{n}} = -\frac{\partial}{\partial r}$ for the normal derivative.

Note that conditions 2 and 3 replace the Neumann data condition on ∂D that we had for imaging a void. $u(r,\theta)$ has the same form as it had for imaging voids:

$$u(r,\theta) = c_0 + d_0 \ln r + \sum_{k\in\mathbb{Z}/\{0\}} (c_k r^{|k|} + d_k r^{-|k|}) e^{ik\theta}$$

From the Dirichlet Boundary Condition $u(1,\theta) = f(\theta)$, we have

$$c_0 + \sum_{k\in\mathbb{Z}/\{0\}} (c_k + d_k) e^{ik\theta} = f(\theta)$$

As we did for a void, we assume f is continuous and piecewise differentiable so that the Fourier series converges pointwise to f. Expanding f as a Fourier Series

$$f(\theta) = \sum_{k\in\mathbb{Z}} f_k e^{ik\theta} \quad f_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta) e^{ik\theta}$$

We find our first expression for the coefficients of u:

$$c_0 = f_0 \quad c_k + d_k = f_k \text{ for } k \in \mathbb{Z}/\{0\} \tag{1}$$

Since u_0 is defined at $r = 0$, the solution for $\alpha\Delta u_0 = 0$ is simply found by eliminating the terms that have singularities at $r = 0$:

$$u_0(r,\theta) = a_0 + \sum_{k\in\mathbb{Z}/\{0\}} a_k r^{|k|} e^{ik\theta}$$

Using the smoothness condition $u_0(\rho,\theta) = u(\rho,\theta)$, we find

$$a_0 = c_0 + d_0 \ln(\rho) \tag{2}$$

$$c_k \rho^{|k|} + d_k \rho^{-|k|} = a_k \rho^{|k|} \text{ for } k \in \mathbb{Z}/\{0\} \tag{3}$$

Using the flux condition $\frac{\partial u}{\partial r}(\rho,\theta) = \alpha \frac{\partial u_0}{\partial r}(\rho,\theta)$, we find

$$d_0 = 0 \tag{4}$$

$$\alpha a_k \rho^{|k|-1} = c_k \rho^{|k|-1} - d_k \rho^{-|k|-1} \tag{5}$$

Using (1), (2), and (4), we find $a_0 = c_0 = f_0$ and $d_0 = 0$. We may employ (1), (3), and (5) to obtain the coefficients a_k, c_k, and d_k. We find

$$a_k = \frac{2f_k}{1+\alpha+(1-\alpha)\rho^{2|k|}} \text{ for } \mathbb{Z}/\{0\} \tag{6}$$

$$c_k = \frac{(1+\alpha)f_k}{1+\alpha+(1-\alpha)\rho^{2|k|}} \text{ for } \mathbb{Z}/\{0\} \tag{7}$$

$$d_k = \frac{(1-\alpha)f_k\rho^{2|k|}}{1+\alpha+(1-\alpha)\rho^{2|k|}} \text{ for } \mathbb{Z}/\{0\} \tag{8}$$

Thus, we have

$$u_0(r,\theta) = f_0 + \sum_{k\in\mathbb{Z}/\{0\}} \frac{2f_k r^{|k|} e^{ik\theta}}{1+\alpha+(1-\alpha)\rho^{2|k|}} \tag{9}$$

$$u(r,\theta) = f_0 + \sum_{k\in\mathbb{Z}/\{0\}} \Big(\frac{(1+\alpha)f_k r^{|k|}}{1+\alpha+(1-\alpha)\rho^{2|k|}} + \frac{(1-\alpha)f_k\rho^{2|k|}r^{-|k|}}{1+\alpha+(1-\alpha)\rho^{2|k|}}\Big)e^{ik\theta} \tag{10}$$

As we did in the Bad Cloaking section, we consider the error in this cloak. Let u_1 be the solution to the Laplacian on Ω with Dirichlet data $u_1 = f$ on $\partial\Omega$ (when Ω is empty). We compute the difference between the Neumann data for u and u_1 (note that u_1 resembles u_0):

$$\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1,\theta) = \sum_{k\in\mathbb{Z}} \frac{2(\alpha-1)|k|\rho^{2|k|}f_k}{1+\alpha+(1-\alpha)\rho^{2|k|}} e^{ik\theta}$$

using the fact that $\frac{\partial}{\partial \mathbf{n}} = \frac{\partial}{\partial r}$ on the boundary $r=1$. Now, computing the $L^2(\partial\Omega)$ norm as

$$\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1,\theta)\Big|\Big|^2_{L^2(\partial\Omega)} = \int_0^{2\pi} \Big|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1,\theta)\Big|^2 d\theta$$

Using Parseval's Identity, we find

$$\begin{aligned}\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1,\theta)\Big|\Big|^2_{L^2(\partial\Omega)} &= \frac{4k^2\rho^{4|k|}}{(1+\alpha+(1-\alpha)\rho^{2|k|})^2}|f_k|^2 \\ &\leq 4k^2\rho^{4|k|}|f_k|^2 \\ &\leq 4\rho^4\Big|\Big|\frac{\partial u_1}{\partial \mathbf{n}}(1,\theta)\Big|\Big|^2_{L^2(\partial\Omega)}\end{aligned}$$

The second line follows from the fact that for $0<\rho<1$, $(1+\alpha+(1-\alpha)\rho^{2|k|})^2 > 1$.
The third line follows from the $L^2(\partial\Omega)$ norm of $u_1(r,\theta)$ and that for $0<\rho<1$ and $k \geq 1$, $\rho^{4|k|} \leq \rho^4$
Finally, taking the square root, we find that the error in the cloaking is proportional to ρ^2 as it was for imaging a void.

$$\Big|\Big|\frac{\partial u}{\partial \mathbf{n}}(1,\theta) - \frac{\partial u_1}{\partial \mathbf{n}}(1,\theta)\Big|\Big|_{L^2(\partial\Omega)} \leq 2\rho^2\Big|\Big|\frac{\partial u_1}{\partial \mathbf{n}}(1,\theta)\Big|\Big|_{L^2(\partial\Omega)} \quad (11)$$

3 Cloaking Conductors

Here we adapt the process from cloaking a void to cloak the region of isotropic conductivity α. Let Ω_ρ be the open annulus $\Omega/\overline{B_\rho(0)}$ with $\rho \in (0,1/3)$ and let u be a twice continuously differentiable solution to the Laplacian on Ω_ρ with $u=f$ on $\partial\Omega_\rho$ and $\frac{\partial u}{\partial r}(\rho,\theta) = \alpha\frac{\partial u_0}{\partial r}(\rho,\theta)$ and $u_0(\rho,\theta) = u(\rho,\theta)$ where u_0 is the solution to the Laplacian on $B_\rho(0)$
We consider the invertible mapping $\phi : \Omega_\rho \to \Omega_{1/3}$ where ϕ and ϕ' are twice continuously differentiable. ϕ will map points $\vec{x} = (x_1,x_2)$ in Ω_ρ to $\vec{y} = (y_1,y_2)$ in $\Omega_{1/3}$ such that for $||\vec{x}|| = \rho$, we have $||\vec{y}|| = 1/3$ and for $||\vec{x}|| = 1$, $||\vec{y}|| = 1$. Lemma 3.1 still holds, that is, for a twice continuously differentiable function v on $\Omega_{1/3}$ (where $v(\vec{y}) = u(\phi^{-1}(\vec{y}))$,

$$\nabla \cdot \sigma(\vec{y})\nabla v = 0$$

in $\Omega_{1/3}$ where $\sigma(\vec{y}) = [D(\phi(\vec{x})(D\phi(\vec{x})^T]/|\det(D\phi(\vec{x})|$ at $\vec{x} = \phi^{-1}(\vec{y})$ where the nonsingular matrix $D\phi(\vec{x})$ is defined as

$$D\phi(\vec{x}) = \begin{bmatrix} \partial y_1/\partial x_1 & \partial y_1/\partial x_2 \\ \partial y_2/\partial x_1 & \partial y_2/\partial x_2 \end{bmatrix}$$

We must construct the cloak such that the problem on Ω_ρ is conserved. We have

$$\phi(\vec{x}) = \frac{\psi(||\vec{x}||}{||\vec{x}||}\vec{x}$$

so that $\vec{y} = \phi(\vec{x})$ means $y_1 = \frac{\psi(||\vec{x}||}{||\vec{x}||}x_1$ and $y_2 = \frac{\psi(||\vec{x}||}{||\vec{x}||}x_2$.
The conditions we must consider are as follows:

1. $\psi(\rho) = 1/3$
2. For some $\delta \in (0,2/3)$, $\psi(r) = r$ for $\frac{1}{3}+\delta \leq r \leq 1$

3. ψ is twice continuously differentiable with $\psi'(r) \geq d_0 > 0$ for some d_0 so that ψ will be strictly increasing and invertible.

4. Unlike the problem with the void, we must define ψ on $0 \leq r \leq \rho$

With these conditions, we may construct $\psi(r)$ as follows:

$$\psi(r) = \begin{cases} \frac{1}{3} + (r - \rho) & 0 < r \leq \rho + \epsilon \\ g(r) & \rho + \epsilon < r < \frac{1}{3} + \delta \\ r & \frac{1}{3} + \delta < r < 1 \end{cases}$$

where $\epsilon \in (0, 1/3)$ guarantees that $\psi(r)$ smoothly transitions from $r < \rho$ to $r > \rho$ through $V_\epsilon(\rho)$. Note that the Laplacian is satisfied for $r < \rho$. Additionally, we have $\lim_{\epsilon \to 0^-} \psi(\rho + \epsilon) = \lim_{\epsilon \to 0^+} \psi(\rho + \epsilon)$ $\lim_{\epsilon \to 0^-} \psi'(\rho + \epsilon) = \lim_{\epsilon \to 0^+} \psi'(\rho + \epsilon)$ so the smoothness conditions: $\frac{\partial u}{\partial r}(\rho, \theta) = \alpha \frac{\partial u_0}{\partial r}(\rho, \theta)$ and $u(\rho, \theta) = u_0(\rho, \theta)$ are conserved in this change of variables. As we had in Exercise 3.3, there are several conditions for $g(r)$:

1. $g(\rho + \epsilon) = \frac{1}{3} + \epsilon$

2. $g'(\rho + \epsilon) = 1$

3. $g''(\rho + \epsilon) = 0$

4. $g(\frac{1}{3} + \delta) = \frac{1}{3} + \delta$

5. $g'(\frac{1}{3} + \delta) = 1$

6. $g''(\frac{1}{3} + \delta) = 0$

As proposed in the paper, we may construct $g(r)$ as a fifth degree polynomial of the form $a_5 r^5 + a_4 r^4 + a_3 r^3 + a_2 r^2 + a_1 r + a_0$ and a computer algebra system may be employed with particular values for ϵ and δ.

Behavior in the Cloaking Region

We may also look at the behavior in the inner cloaking region $\frac{1}{3} \leq ||\vec{y}|| \leq \frac{1}{3} + \epsilon$ (corresponding to $\rho \leq ||\vec{x}|| \leq \rho + \epsilon$). We still have the same expression for $D\phi$:

$$D\phi = (\psi'(r)/r^2 + \psi(r)/r^3) \begin{bmatrix} x_1^2 & x_1 x_2 \\ x_1 x_2 & x_2^2 \end{bmatrix} + (\psi(r)/r) I$$

Using $\sigma = (D\phi)^2 / |\det D\phi|$ in conjunction with Exercises 3.4 and 3.5, we find the eigenvectors $\vec{v}_1 = [x_1, x_2]^T$ and $\vec{v}_2 = [-x_2, x_1]^T$, with eigenvalues $\mu_1 = \psi'(r)$ and $\mu_2 = \psi(r)/r$. $\det D\phi = \mu_1 \mu_2$ so for the eigenvalues of σ, we find

$$\gamma_m = \frac{r\psi'(r)}{\psi(r)} \quad \gamma_M = \frac{\psi(r)}{r\psi'(r)}$$

Using the constructed $\psi(r)$:

$$\gamma_m = \frac{3r}{1 + 3r - 3\rho} \quad \gamma_M = \frac{1 + 3r - 3\rho}{3r}$$

Our conclusions regarding these eigenvalues are very similar to those made in the paper. At $r = \rho$, we find

$$\gamma_m = 3\rho \quad \gamma_M = \frac{1}{3\rho}$$

where as $\rho \to 0^+$, $\gamma_m \to 0^+$ and γ_M becomes arbitrarily large. As you can see, despite the difference in the region being one of isotropic conductivity rather than a void, the behavior of the cloak is similar.

4 Conclusion

The paper by Kurt Bryan and Tanya Leise provides a good base understanding of the cloaking problem. The paper covers a simplified example of a PDE for which an explicit solution can be obtained using separation of variables. However, it occurs more often than not that these types of explicit solutions are not available. Nonetheless, the solutions discussed make it easy to study the behavior of the solution in certain regions such as the inner cloaking region. Using the two problems discussed here, cloaking a void and cloaking a region of isotropic conductivity, one could rig up a problem with n concentric circles and certain boundary conditions to cloak some regions as voids and others as regions as ones with a certain isotropic conductivity. The problems could be tedious but much of the work will be the same. Applying the Dirichlet data condition $u = f$ on $\partial\Omega$ and either the Neumann data condition $\frac{\partial u}{\partial \mathbf{n}} = 0$ or a flux condition $\alpha_k \frac{\partial u_k}{\partial \mathbf{n}} = \alpha_{k-1} \frac{\partial u_{k-1}}{\partial \mathbf{n}}$ (where α_k and α_{k-1} denotes the isotropic conductivity in regions corresponding to solutions u_k and u_{k-1}) and solving the Laplacian $\Delta u = 0$ will yield certain functions in each notable region. The whole point is that no matter what is placed inside $B_\rho(0)$ and whatever is the PDE used to describe it, the method to obtain the cloaking layer is the same. The layer of anisotropic conductivity will cause the current to flow past the ball as if nothing is there to the order of ρ^2.